Advanced composites in engineering structures

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Lecture #4

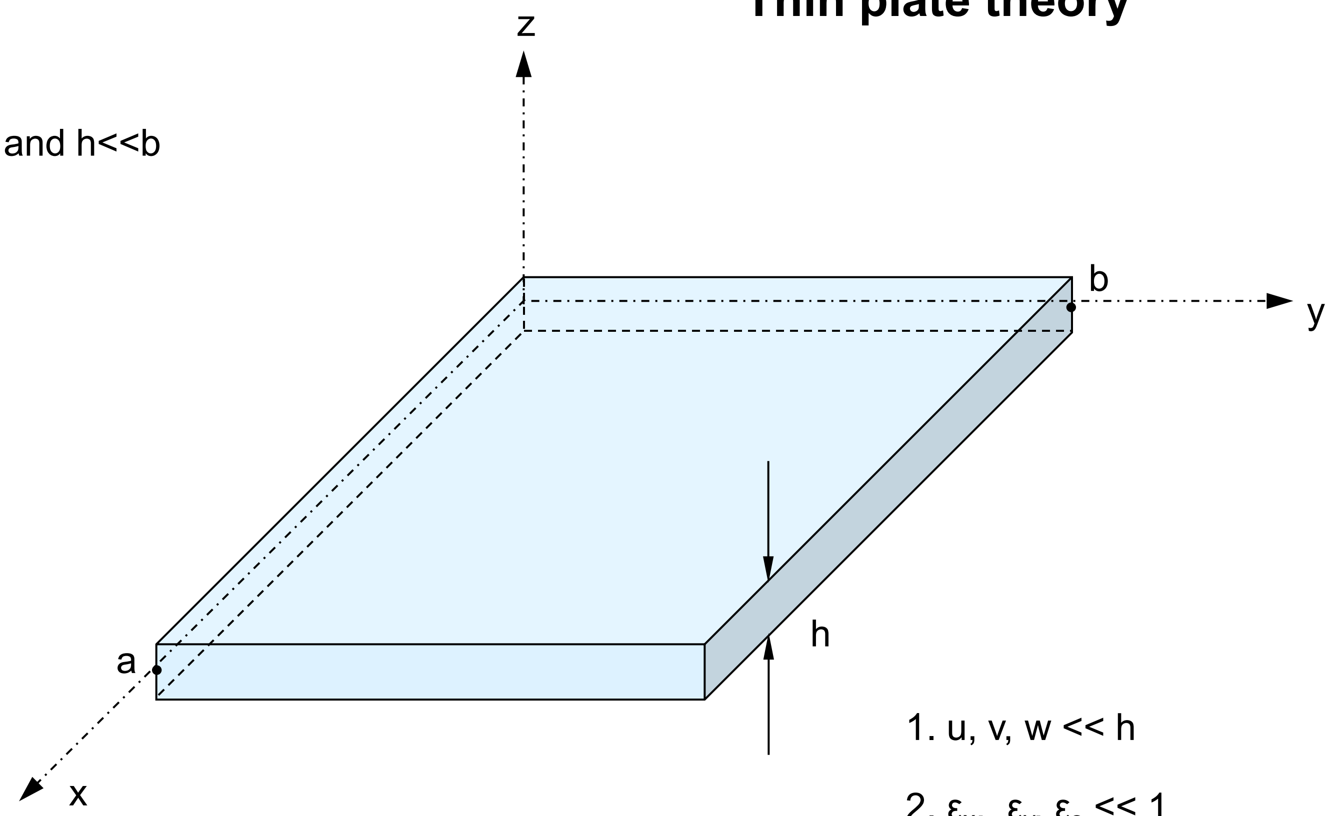
Analysis of laminates

Plane stress assumption

- Consider a thin plate without out-of-plane loads, then $\sigma_3=0$, $\tau_{13}=\tau_{23}=0$.
- In this case we can assume that these stresses are also zero inside the (THIN) plate.
- Then, the three-dimensional stress-strain equations can be reduced to two dimensional.

Thin plate theory

h<<a and h<<b



1. u, v, w << h
2. ε_x, ε_y, ε_s << 1
3. ε_{xz}, ε_{yz} negligible
4. u, v linear functions of the thickness, z
5. ε_z negligible

Assumptions for deformations and strains:

6. The plate reacts (accepts) lateral or in-plane loads and develops in-plane and stresses (x-y plane)
(No out of plane loads)

Therefore, the only non-zero strain tensor components are

$$\begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} & \cancel{\varepsilon_{xz}} \\ \varepsilon_{xy} & \varepsilon_{yy} & \cancel{\varepsilon_{yz}} \\ \cancel{\varepsilon_{xz}} & \cancel{\varepsilon_{yz}} & \cancel{\varepsilon_{zz}} \end{bmatrix} \rightarrow \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_s \end{Bmatrix} \rightarrow$$

$$u = u_o(x,y) - z\frac{\partial w}{\partial x}$$

$$v = v_o(x,y) - z\frac{\partial w}{\partial y}$$

$$w = w_o(x,y)$$

$$\varepsilon_x = \frac{\partial u}{\partial x} = \frac{\partial u_o}{\partial x} - z\frac{\partial^2 w}{\partial x^2}$$

$$\varepsilon_y = \frac{\partial v}{\partial y} = \frac{\partial v_o}{\partial y} - z\frac{\partial^2 w}{\partial y^2}$$

$$\varepsilon_s = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = \frac{\partial u_o}{\partial y} + \frac{\partial v_o}{\partial x} - 2z\frac{\partial^2 w}{\partial x \partial y}$$

$$\varepsilon_x^o = \frac{\partial u_o(x,y)}{\partial x}, \quad \varepsilon_y^o = \frac{\partial v_o(x,y)}{\partial y}, \quad \varepsilon_s^o = \frac{\partial u_o(x,y)}{\partial y} + \frac{\partial v_o(x,y)}{\partial x}$$

$$k_x = -\frac{\partial^2 w}{\partial x^2}, \quad k_y = -\frac{\partial^2 w}{\partial y^2}, \quad k_s = -2\frac{\partial^2 w}{\partial x \partial y}$$

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_s \end{Bmatrix} = \begin{Bmatrix} \varepsilon_x^o \\ \varepsilon_y^o \\ \varepsilon_s^o \end{Bmatrix} + z \begin{Bmatrix} k_x \\ k_y \\ k_s \end{Bmatrix}$$

$$\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}^{\mathbf{o}} + z\mathbf{k}$$

k_x: Curvature at x direction=$1/\rho_x$

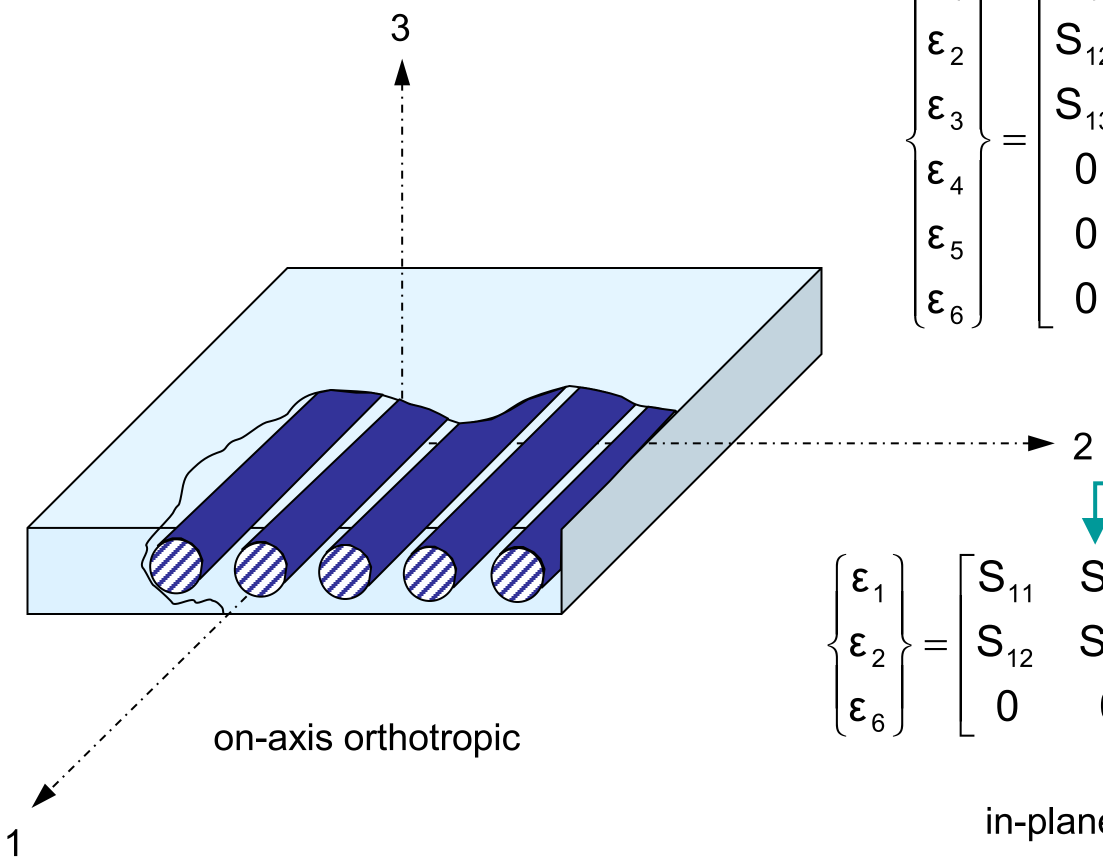
k_y: Curvature at y direction =$1/\rho_y$

k_s: Twisting

Lamina (thin layer) of orthotropic material: $\boxed{\text{plain stress}}$

$$\sigma_3 = \sigma_4 = \sigma_5 = 0$$

$$\begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \end{Bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{13} & 0 & 0 & 0 \\ S_{12} & S_{22} & S_{23} & 0 & 0 & 0 \\ S_{13} & S_{23} & S_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & S_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & S_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & S_{66} \end{bmatrix} \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ 0 \\ 0 \\ 0 \\ \sigma_6 \end{Bmatrix}$$



on-axis orthotropic

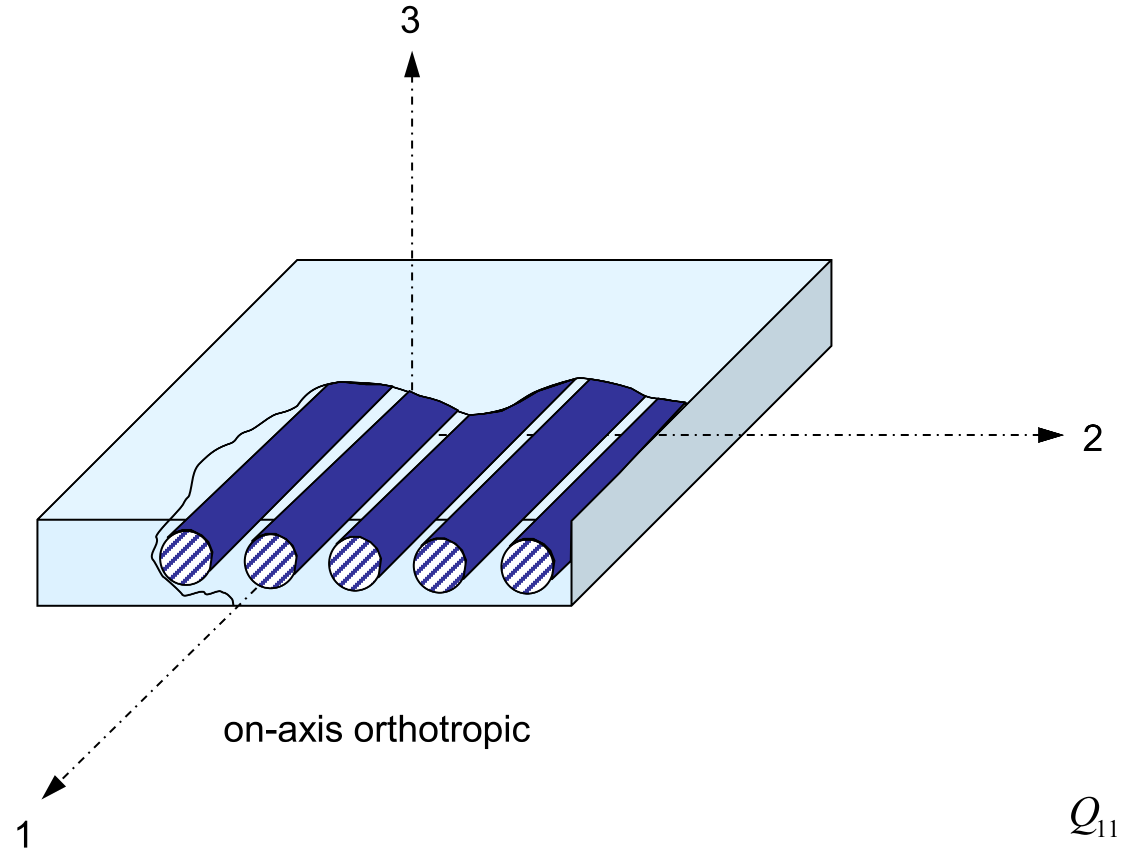
$$\begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_6 \end{Bmatrix} = \begin{bmatrix} S_{11} & S_{12} & 0 \\ S_{12} & S_{22} & 0 \\ 0 & 0 & S_{66} \end{bmatrix} \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_6 \end{Bmatrix}$$

in-plane

$$\varepsilon_3 = S_{31}\sigma_1 + S_{32}\sigma_2$$
$$\varepsilon_4 = \varepsilon_5 = 0$$

out-of-plane

Lamina (thin layer) of orthotropic material: $\boxed{\text{plain stress}}$



$$\begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_6 \end{Bmatrix} = \begin{bmatrix} S_{11} & S_{12} & 0 \\ S_{12} & S_{22} & 0 \\ 0 & 0 & S_{66} \end{bmatrix} \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_6 \end{Bmatrix}$$

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_6 \end{Bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{12} & Q_{22} & 0 \\ 0 & 0 & Q_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_6 \end{Bmatrix}$$

Q: reduced stiffness matrix

$$Q_{11} = \frac{E_1}{1-\nu_{12}\nu_{21}}, \quad Q_{22} = \frac{E_2}{1-\nu_{12}\nu_{21}}, \quad Q_{12} = \frac{\nu_{12}E_2}{1-\nu_{12}\nu_{21}} =_1 \frac{\nu_{21}E}{1-\nu_{12}\nu_{21}}$$

$$Q_{66} = G_{12}$$

$$Q_{11} = S_{22}D^{-1}, Q_{22} = S_{11}D^{-1}, Q_{12} = -S_{12}D^{-1}$$

$$Q_{66} = S_{66}^{-1}, D = S_{11}S_{22} - S_{12}^2$$

Material's Constants

$$\boxed{Q_{11}} = \frac{S_{22}}{S_{11}S_{22} - S_{12}^{\ 2}}$$

$$\boxed{Q_{12}} = -\frac{S_{12}}{S_{11}S_{22} - S_{12}^{\ 2}}$$

$$\boxed{Q_{22}} = \frac{S_{11}}{S_{11}S_{22} - S_{12}^{\ 2}}$$

$$\boxed{Q_{66}} = \frac{1}{S_{66}}$$

$$\boxed{S_{11}} = \frac{1}{E_1}$$

$$\boxed{S_{12}} = -\frac{\nu_{12}}{E_1}$$

$$\boxed{S_{22}} = \frac{1}{E_2}$$

$$\boxed{S_{66}} = \frac{1}{G_{12}}$$

$$\boxed{Q_{11}} = \frac{E_1}{1 - \nu_{21}\nu_{12}}$$

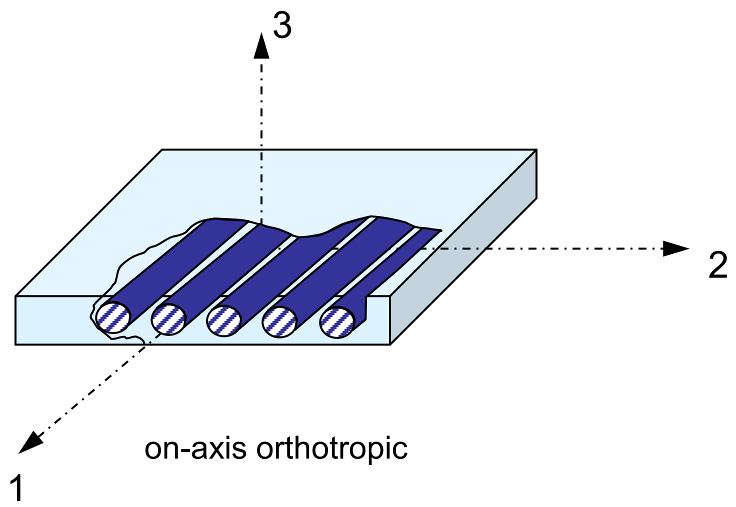
$$\boxed{Q_{12}} = \frac{\nu_{12}E_2}{1 - \nu_{21}\nu_{12}}$$

$$\boxed{Q_{22}} = \frac{E_2}{1 - \nu_{21}\nu_{12}}$$

$$\boxed{Q_{66}} = G_{12}$$

UD lamina is a special case of orthotropic lamina, since normal stresses applied in the 1-2 direction do not result in any shearing strains in the 1-2 plane. This happens because $Q_{16}=Q_{26}=S_{16}=S_{26}=0$. This comment also applies for woven composites.

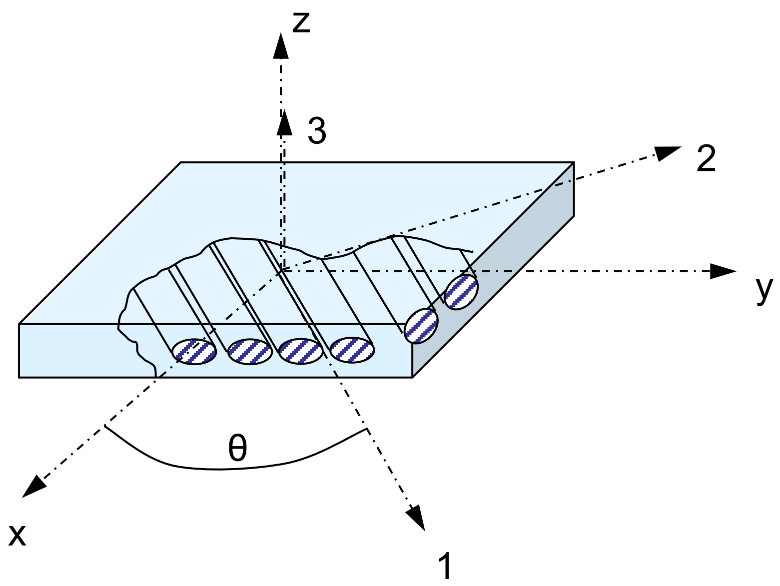
Plane stress – off axis



on-axis orthotropic

$$\begin{Bmatrix}\varepsilon_1\\ \varepsilon_2\\ \varepsilon_6\end{Bmatrix} = \begin{bmatrix}S_{11} & S_{12} & 0\\ S_{12} & S_{22} & 0\\ 0 & 0 & S_{66}\end{bmatrix}\begin{Bmatrix}\sigma_1\\ \sigma_2\\ \sigma_6\end{Bmatrix}$$

$$\begin{Bmatrix}\sigma_1\\ \sigma_2\\ \sigma_6\end{Bmatrix} = \begin{bmatrix}Q_{11} & Q_{12} & 0\\ Q_{12} & Q_{22} & 0\\ 0 & 0 & Q_{66}\end{bmatrix}\begin{Bmatrix}\varepsilon_1\\ \varepsilon_2\\ \varepsilon_6\end{Bmatrix}$$

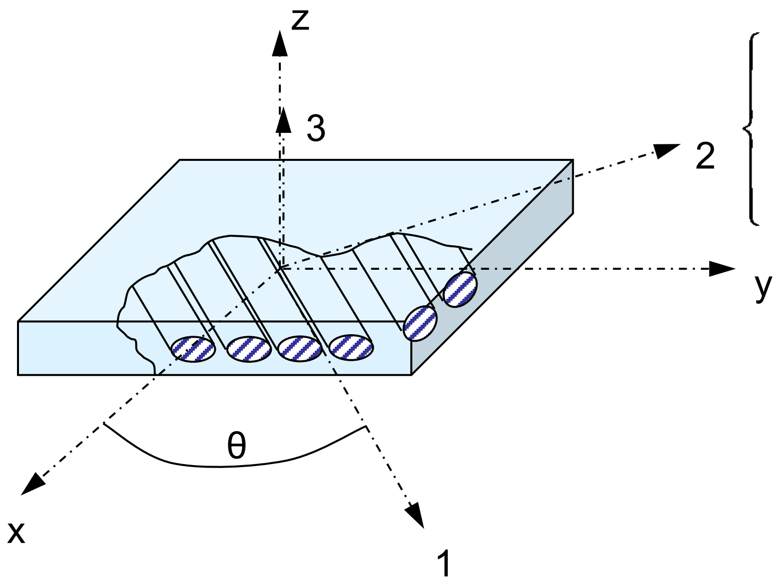


off-axis orthotropic

$$\begin{Bmatrix}\varepsilon_x\\ \varepsilon_y\\ \varepsilon_s\end{Bmatrix} = \begin{bmatrix}S_{xx} & S_{xy} & S_{xs}\\ S_{yx} & S_{yy} & S_{ys}\\ S_{sx} & S_{sy} & S_{ss}\end{bmatrix}\begin{Bmatrix}\sigma_x\\ \sigma_y\\ \sigma_s\end{Bmatrix}$$

$$\begin{Bmatrix}\sigma_x\\ \sigma_y\\ \sigma_s\end{Bmatrix} = \begin{bmatrix}Q_{xx} & Q_{xy} & Q_{xs}\\ Q_{yx} & Q_{yy} & Q_{ys}\\ Q_{sx} & Q_{sy} & Q_{ss}\end{bmatrix}\begin{Bmatrix}\varepsilon_x\\ \varepsilon_y\\ \varepsilon_s\end{Bmatrix}$$

Off-axis or general orthotropic lamina



$$\begin{Bmatrix}\varepsilon_x\\ \varepsilon_y\\ \varepsilon_s\end{Bmatrix}=\begin{bmatrix}S_{xx} & S_{xy} & S_{xs}\\ S_{yx} & S_{yy} & S_{ys}\\ S_{sx} & S_{sy} & S_{ss}\end{bmatrix}\begin{Bmatrix}\sigma_x\\ \sigma_y\\ \sigma_s\end{Bmatrix} \quad \text{or} \quad \begin{Bmatrix}\sigma_x\\ \sigma_y\\ \sigma_s\end{Bmatrix}=\begin{bmatrix}Q_{xx} & Q_{xy} & Q_{xs}\\ Q_{yx} & Q_{yy} & Q_{ys}\\ Q_{sx} & Q_{sy} & Q_{ss}\end{bmatrix}\begin{Bmatrix}\varepsilon_x\\ \varepsilon_y\\ \varepsilon_s\end{Bmatrix}$$

$$S'_{11}=S_{11}m^4+m^2n^2\left(2S_{12}+S_{66}\right)+S_{22}n^4$$

$$S'_{12}=m^2n^2\left(S_{11}+S_{22}-S_{66}\right)+S_{12}\left(m^4+n^4\right)$$

$$S'_{13}=S_{13}m^2+S_{23}n^2$$

$$S'_{16}=mn\left[2S_{11}m^2-2S_{22}n^2-\left(2S_{12}+S_{66}\right)\left(m^2-n^2\right)\right]$$

$$S'_{22}=S_{11}n^4+m^2n^2\left(2S_{12}+S_{66}\right)+S_{22}m^4$$

$$S'_{23}=S_{13}n^2+S_{23}m^2$$

$$S'_{26}=mn\left[2S_{11}n^2-2S_{22}m^2+\left(2S_{12}+S_{66}\right)\left(m^2-n^2\right)\right]$$

$$S'_{33}=S_{33}$$

$$S'_{36}=2\left(S_{13}-S_{23}\right)mn$$

$$S'_{44}=S_{44}m^2+S_{55}n^2$$

$$S'_{45}=\left(S_{55}-S_{44}\right)mn$$

$$S'_{55}=S_{44}n^2+S_{55}m^2$$

$$S'_{66}=2\left(2S_{11}+2S_{22}-4S_{12}\right)m^2n^2+S_{66}\left(m^2-n^2\right)^2$$

$$S'_{14}=S'_{15}=S'_{24}=S'_{25}=S'_{34}=S'_{35}=S'_{46}=S'_{56}=0$$

$$S_{xx}=S_{11}m^4+m^2n^2\left(2S_{12}+S_{66}\right)+S_{22}n^4$$

$$S_{xy}=m^2n^2\left(S_{11}+S_{22}-S_{66}\right)+S_{12}\left(m^4+n^4\right)$$

$$S_{xs}=mn\left[2S_{11}m^2-2S_{22}n^2-\left(2S_{12}+S_{66}\right)\left(m^2-n^2\right)\right]$$

$$S_{yy}=S_{11}n^4+m^2n^2\left(2S_{12}+S_{66}\right)+S_{22}m^4$$

$$S_{ys}=mn\left[2S_{11}n^2-2S_{22}m^2+\left(2S_{12}+S_{66}\right)\left(m^2-n^2\right)\right]$$

$$S_{ss}=2\left(2S_{11}+2S_{22}-4S_{12}\right)m^2n^2+S_{66}\left(m^2-n^2\right)^2$$

Same for the stiffness matrix

$$C'_{11} = C_{11}m^4 + 2m^2n^2(C_{12} + 2C_{66}) + C_{22}n^4$$
$$C'_{12} = m^2n^2(C_{11} + C_{22} - 4C_{66}) + C_{12}(m^4 + n^4)$$
$$C'_{13} = C_{13}m^2 + C_{23}n^2$$
$$C'_{16} = mn\left[C_{11}m^2 - C_{22}n^2 - (C_{12} + 2C_{66})(m^2 - n^2)\right]$$
$$C'_{22} = C_{11}n^4 + 2m^2n^2(C_{12} + 2C_{66}) + C_{22}m^4$$
$$C'_{23} = C_{13}n^2 + C_{23}m^2$$
$$C'_{26} = mn\left[C_{11}n^2 - C_{22}m^2 + (C_{12} + 2C_{66})(m^2 - n^2)\right]$$
$$C'_{33} = C_{33}$$
$$C'_{36} = (C_{13} - C_{23})mn$$
$$C'_{44} = C_{44}m^2 + C_{55}n^2$$
$$C'_{45} = (C_{55} - C_{44})mn$$
$$C'_{55} = C_{44}n^2 + C_{55}m^2$$
$$C'_{66} = (C_{11} + C_{22} - 2C_{12})m^2n^2 + C_{66}(m^2 - n^2)^2$$
$$C'_{14} = C'_{15} = C'_{24} = C'_{25} = C'_{34} = C'_{35} = C'_{46} = C'_{56} = 0$$

$$Q_{xx} = Q_{11}m^4 + 2(Q_{12} + 2Q_{66})m^2n^2 + Q_{22}n^4$$
$$Q_{xy} = (Q_{11} + Q_{22} - 4Q_{66})m^2n^2 + Q_{12}(m^4 + n^4)$$
$$Q_{yy} = Q_{11}n^4 + 2(Q_{12} + 2Q_{66})m^2n^2 + Q_{22}m^4$$
$$Q_{xs} = (Q_{11} - Q_{12} - 2Q_{66})nm^3 + (Q_{12} - Q_{22} + 2Q_{66})n^3m$$
$$Q_{ys} = (Q_{11} - Q_{12} - 2Q_{66})mn^3 + (Q_{12} - Q_{22} + 2Q_{66})m^3n$$
$$Q_{ss} = (Q_{11} + Q_{22} - 2Q_{12} - 2Q_{66})n^2m^2 + Q_{66}(n^4 + m^4)$$

Transformed reduced compliance matrix

From the previous equations referring to the unidirectional lamina loaded in the material axes directions (on-axis), no coupling occurs between the normal and shearing terms of stresses and strains.

However, for an angle lamina (off-axis), as indicated in these last equations, coupling takes place between the normal and shearing components of stresses and strains.

Thus, if only normal stresses are applied to an angle lamina, then the shear strains are non-zero and vise-versa if shear stresses are applied, the normal strains are non-zero.

That is why the angle ply lamina is called **Generally Orthotropic Lamina**.

Engineering constants of the off-axis lamina

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_s \end{Bmatrix} = \begin{bmatrix} S_{xx} & S_{xy} & S_{xs} \\ S_{yx} & S_{yy} & S_{ys} \\ S_{sx} & S_{sy} & S_{ss} \end{bmatrix} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_s \end{Bmatrix} \longrightarrow [S_{ij}] = \begin{bmatrix} \frac{1}{E_x} & \frac{-\nu_{yx}}{E_y} & \frac{m_{sx}}{G_{xy}} \\ \frac{-\nu_{xy}}{E_x} & \frac{1}{E_y} & \frac{m_{sy}}{G_{xy}} \\ \frac{m_{xs}}{E_x} & \frac{m_{ys}}{E_y} & \frac{1}{G_{xy}} \end{bmatrix}$$

Finally, six constants.

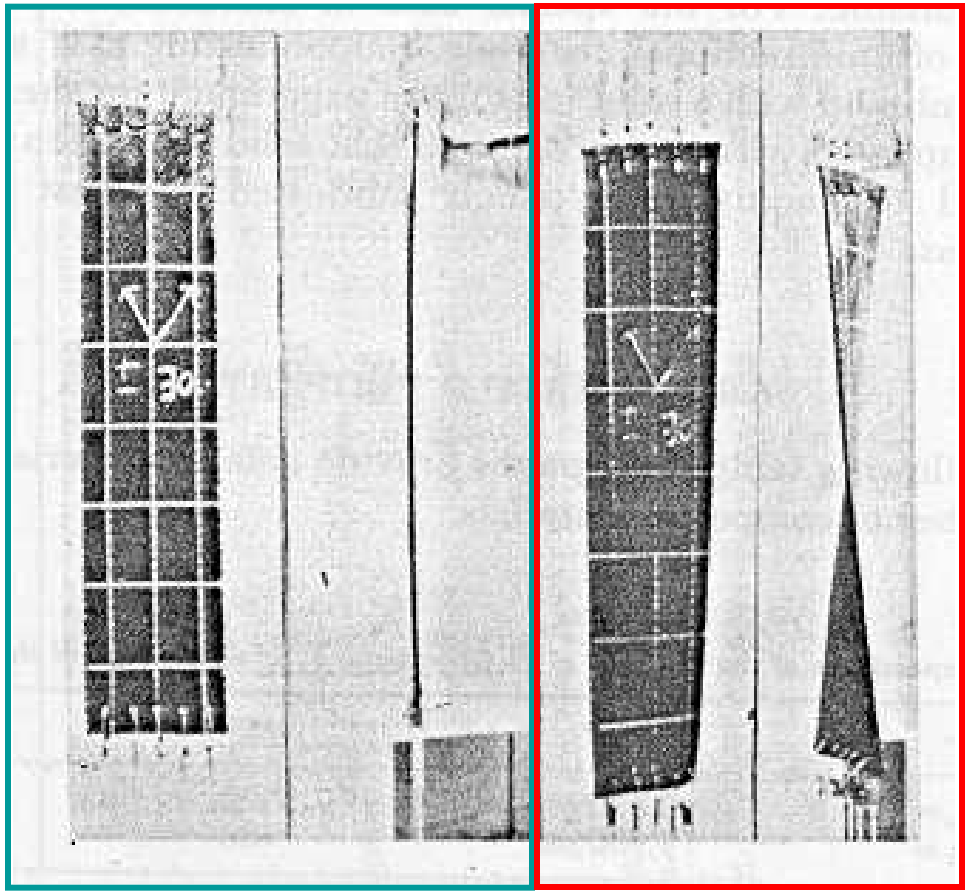
However, these **six** constants can be expressed as functions of the four engineering constants of the unidirectional ply; $\mathbf{E_1}$, $\mathbf{E_2}$, $\mathbf{v_{12}}$, $\mathbf{G_{12}}$. (**4** and not **5** since we are in 2D)

m_{is}, m_{si}, i=x,y: Coefficients of shear coupling (Chentsov coefficients)

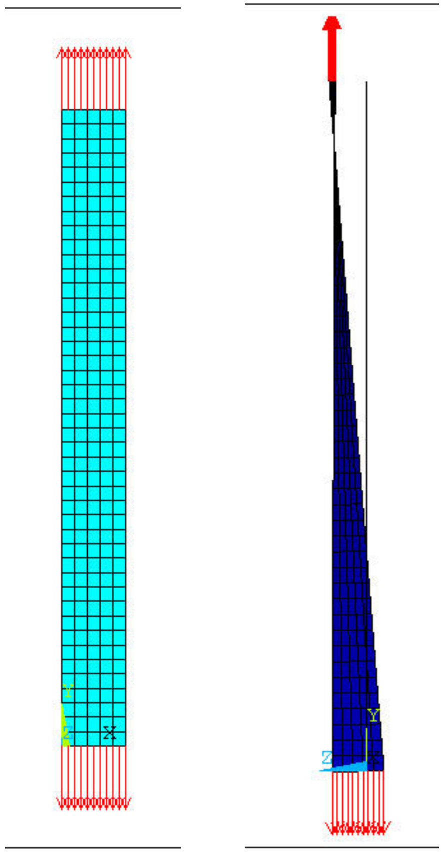
$m_{si} = \dfrac{e_i}{e_s}$: Is the ratio between the normal strain along the i-direction, caused by shear stress σ_s, over the corresponding shear strain.

$m_{is} = \dfrac{e_s}{e_i}$: Is the ratio of the shear strain over the normal along i-direction due to the normal stress along the same direction.

An angle-ply, balanced, asymmetric laminate presents coupling between the normal force and the shear strain

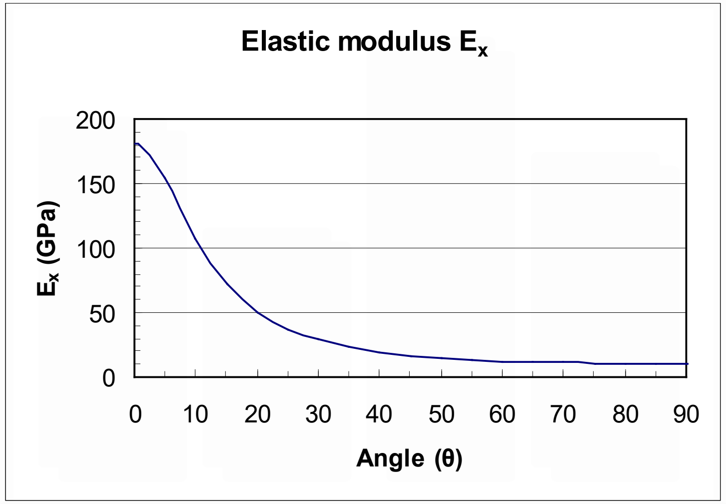


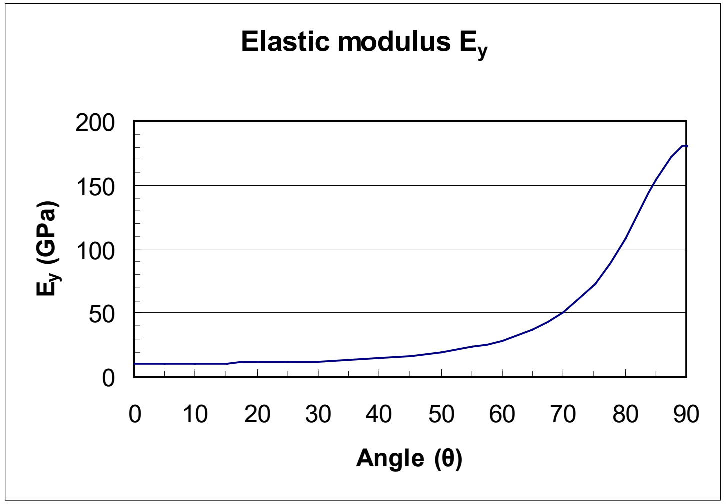
Un-deformed laminate Deformed laminate

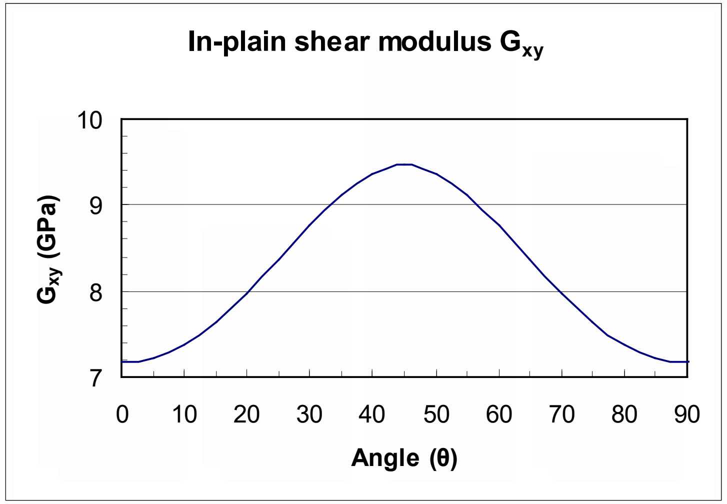


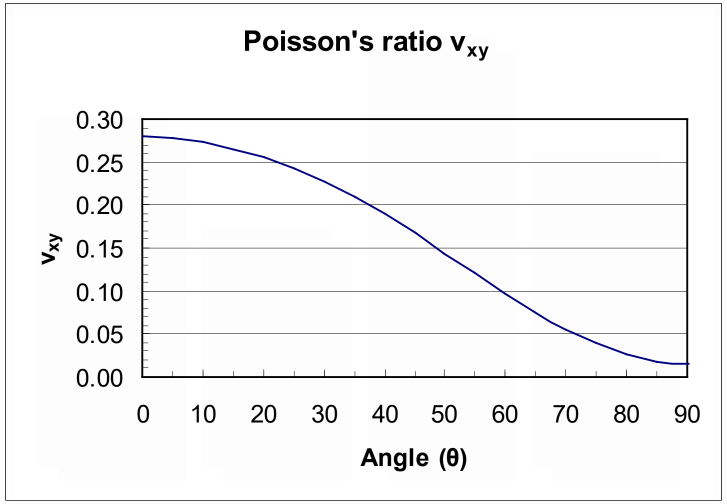
J. E. Ashton, J. C. Halpin, P. H. Petit, “Primer on Composite Materials: Analysis”, Technomic 1969

Elastic constants vs off-axis angle for a graphite/epoxy lamina

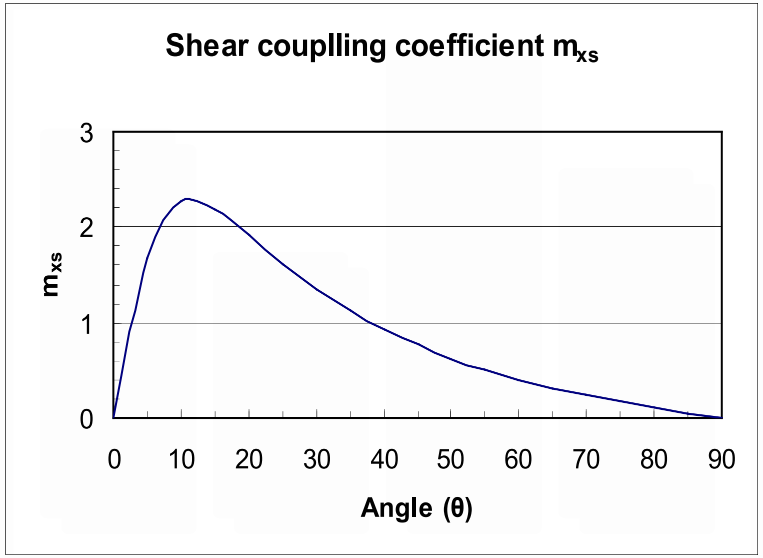


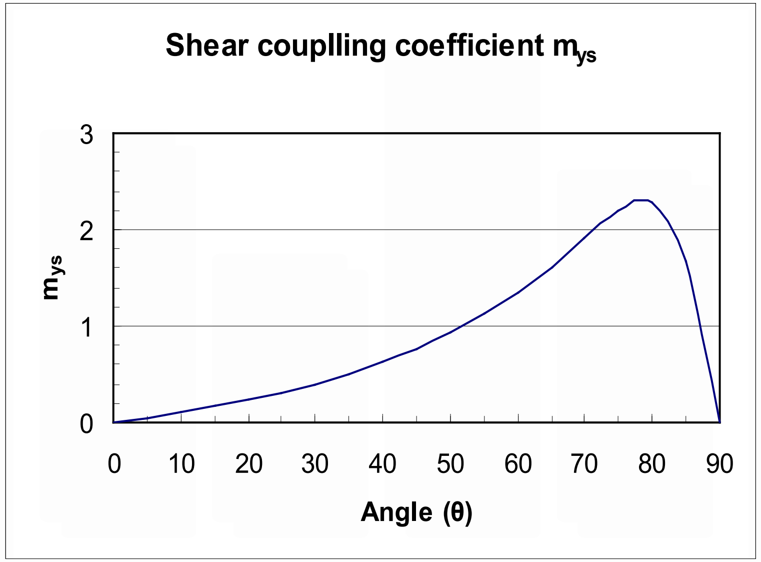






$$E_1 = 181GPa,\ E_2 = 10.3GPa,\ G_{12} = 7.17GPa,\ and\ v_{12} = 0.28$$



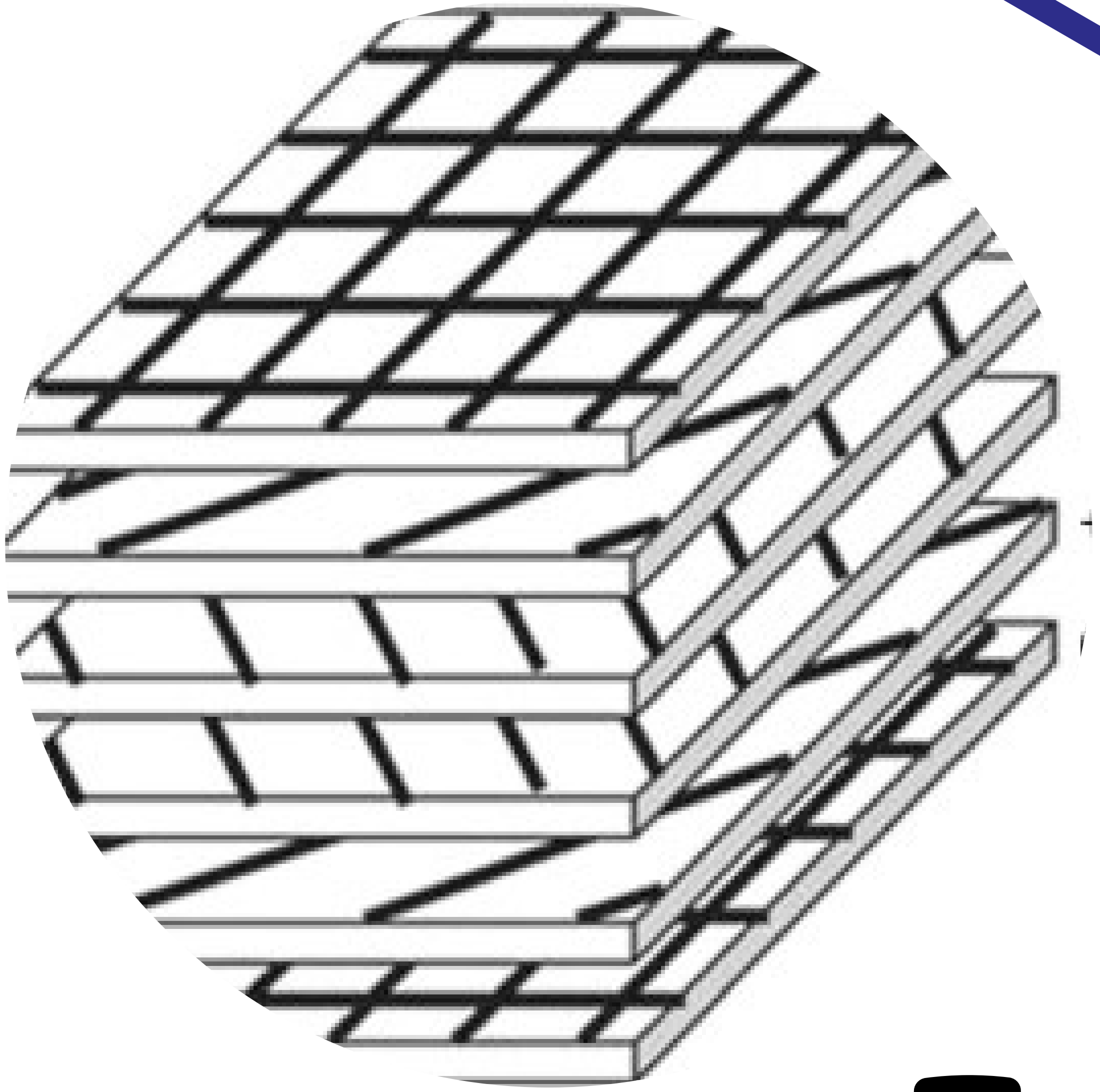


The values of the shear coupling coefficients present the normal-shear coupling in the laminate.

Compared to Poisson's ratio they have extreme values which means that the normal-shear coupling have a stronger effect.

This phenomenon is missing in isotropic materials and unidirectional plies, but as shown herein cannot be ignored in angle plies.

$$E_1 = 181GPa,\ E_2 = 10.3GPa,\ G_{12} = 7.17GPa,\ and\ \nu_{12} = 0.28$$



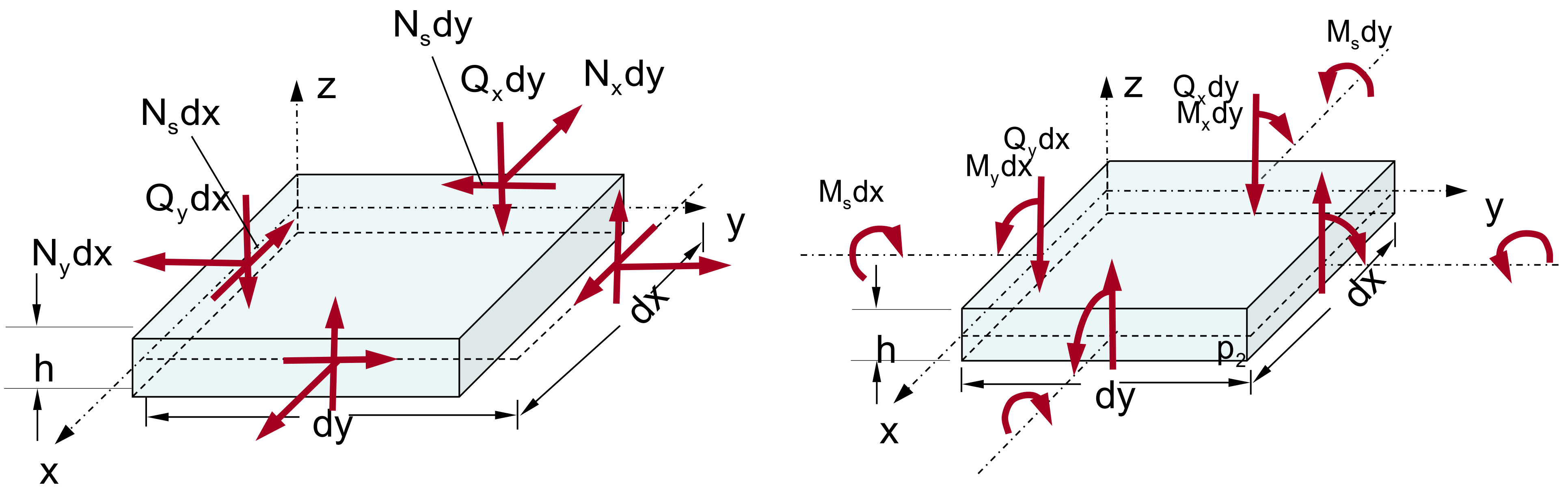
Laminate analysis

Laminate analysis / mechanics of laminates

- OBJECTIVE: Develop relationships between mechanical loads applied to a laminate and stresses/strains in each one of the laminae (or plies or layers) that comprise that laminate.

The problem

- Given external loads on a laminate (tensile/compressive, shear, bending or torsional) determine stress and strain fields on each ply



Force and moment resultants on a laminate

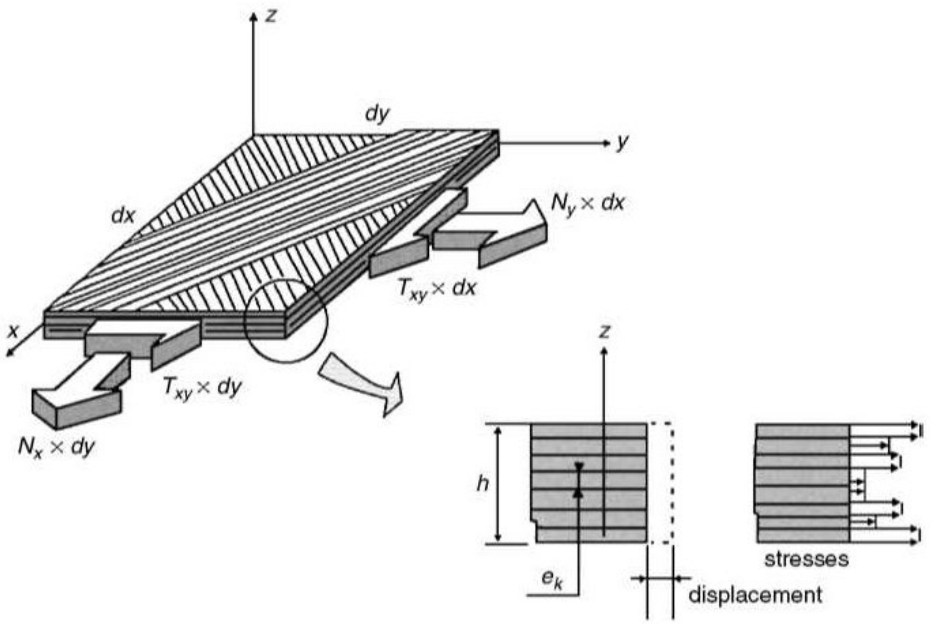
- Force and moment resultants are the normal force, shear force, bending moments and twisting moments that could be applied on a laminate, **PER** unit length.
- The units of force resultants: $Pa\text{-}m$ or N/m
- The units of moment resultants: $Pa\text{-}m^2$ or Nm/m

$$N_x, N_y, N_{xy} \qquad (Pa\text{-}m \text{ or } N/m)$$

$$M_x, M_y, M_{xy} \qquad (Pa\text{-}m^2 \text{ or } Nm/m)$$

Force resultants

It is possible to substitute the stress resultants with the **global average stresses** of the laminate (which are fictitious)



$$\sigma_x = \frac{N_x}{h}$$

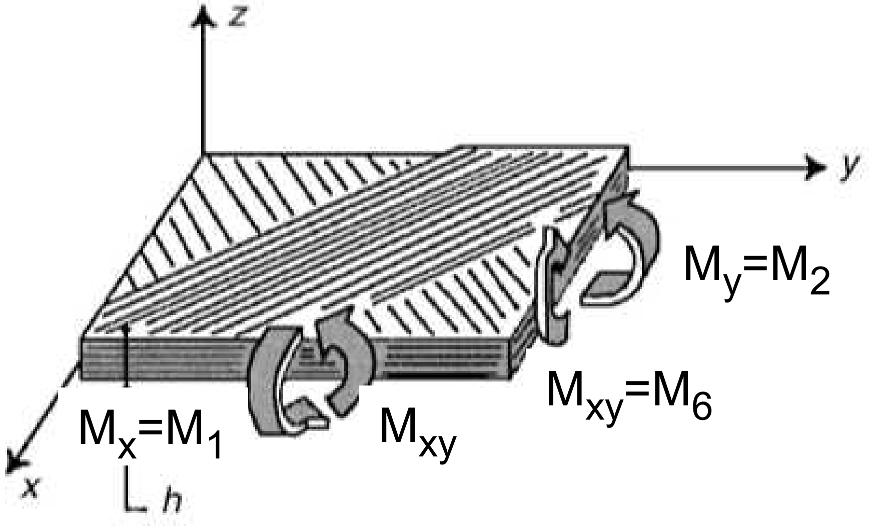
$$\sigma_y = \frac{N_y}{h}$$

$$\sigma_s = \frac{N_s}{h}$$

$$\begin{Bmatrix} N_x \\ N_y \\ N_s \end{Bmatrix} = \int_{-h/2}^{h/2} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_s \end{Bmatrix} dz$$

Moment resultants

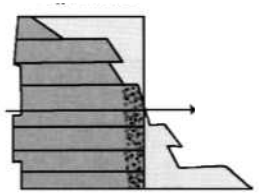
- M_x: The moment resultant along x direction due to the stress σ_y, over a unit width along the x direction



$$M_x = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_x z dz$$

$$M_y = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_y z dz$$

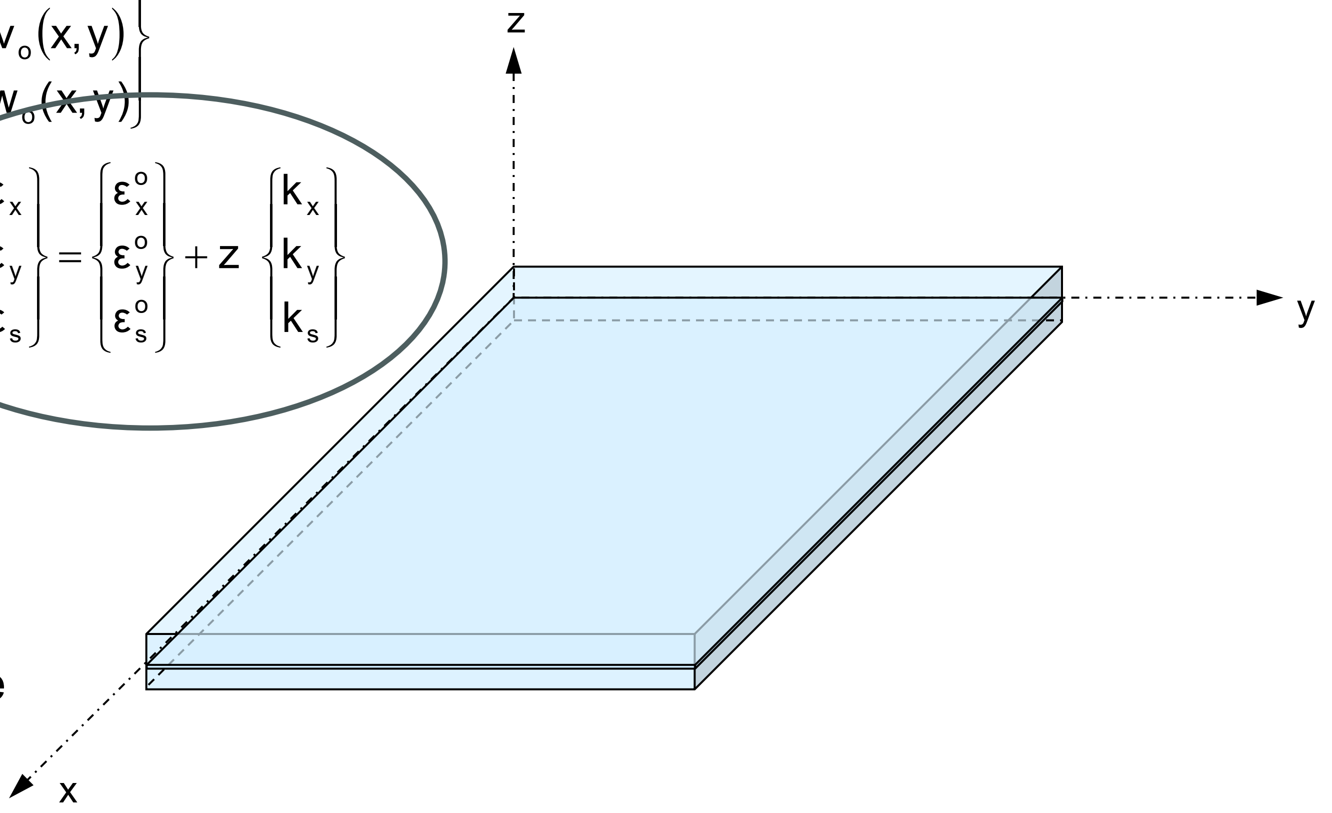
$$M_s = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_s z dz$$



Force and moment resultants at the mid-plane

$$\begin{Bmatrix} u_o(x,y) \\ v_o(x,y) \\ w_o(x,y) \end{Bmatrix}$$

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_s \end{Bmatrix} = \begin{Bmatrix} \varepsilon_x^o \\ \varepsilon_y^o \\ \varepsilon_s^o \end{Bmatrix} + z \begin{Bmatrix} k_x \\ k_y \\ k_s \end{Bmatrix}$$



Homogenization

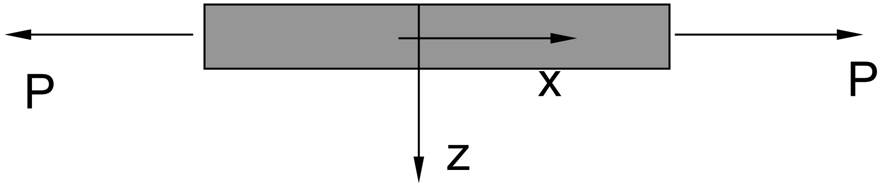
Derive the constitutive equations:

Instead of stresses throughout the laminate thickness, we use an equivalent system of force and moment resultants applied in the laminate mid-plane:

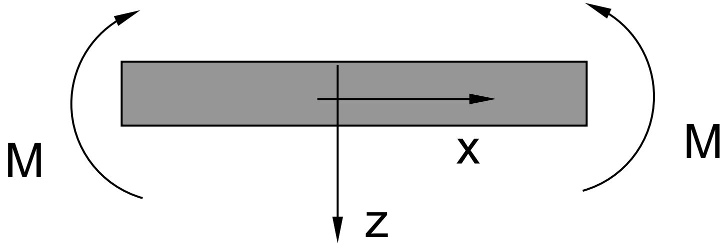
$$\begin{Bmatrix} N_x \\ N_y \\ N_s \end{Bmatrix} = \int_{-h/2}^{h/2} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_s \end{Bmatrix} dz \qquad \begin{Bmatrix} M_x \\ M_y \\ M_s \end{Bmatrix} = \int_{-h/2}^{h/2} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_s \end{Bmatrix} zdz$$

$$\begin{Bmatrix} Q_x \\ Q_y \end{Bmatrix} = \int_{-h/2}^{h/2} \begin{Bmatrix} \sigma_{xz} \\ \sigma_{yz} \end{Bmatrix} dz,$$

Strains for an Isotropic beam

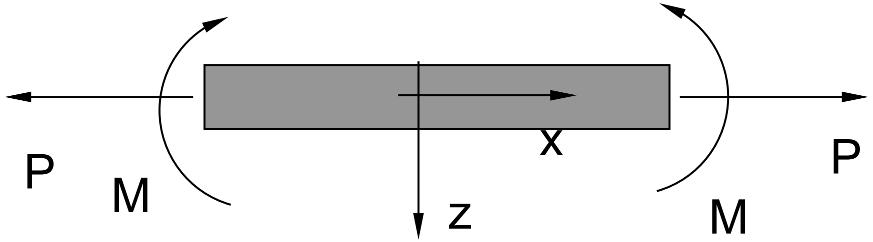


$$\sigma_x = \frac{P}{A} \qquad \varepsilon_x = \frac{P}{EA}$$



$$\sigma_x = \frac{M}{I} z \qquad \varepsilon_x = \frac{M}{EI} z$$

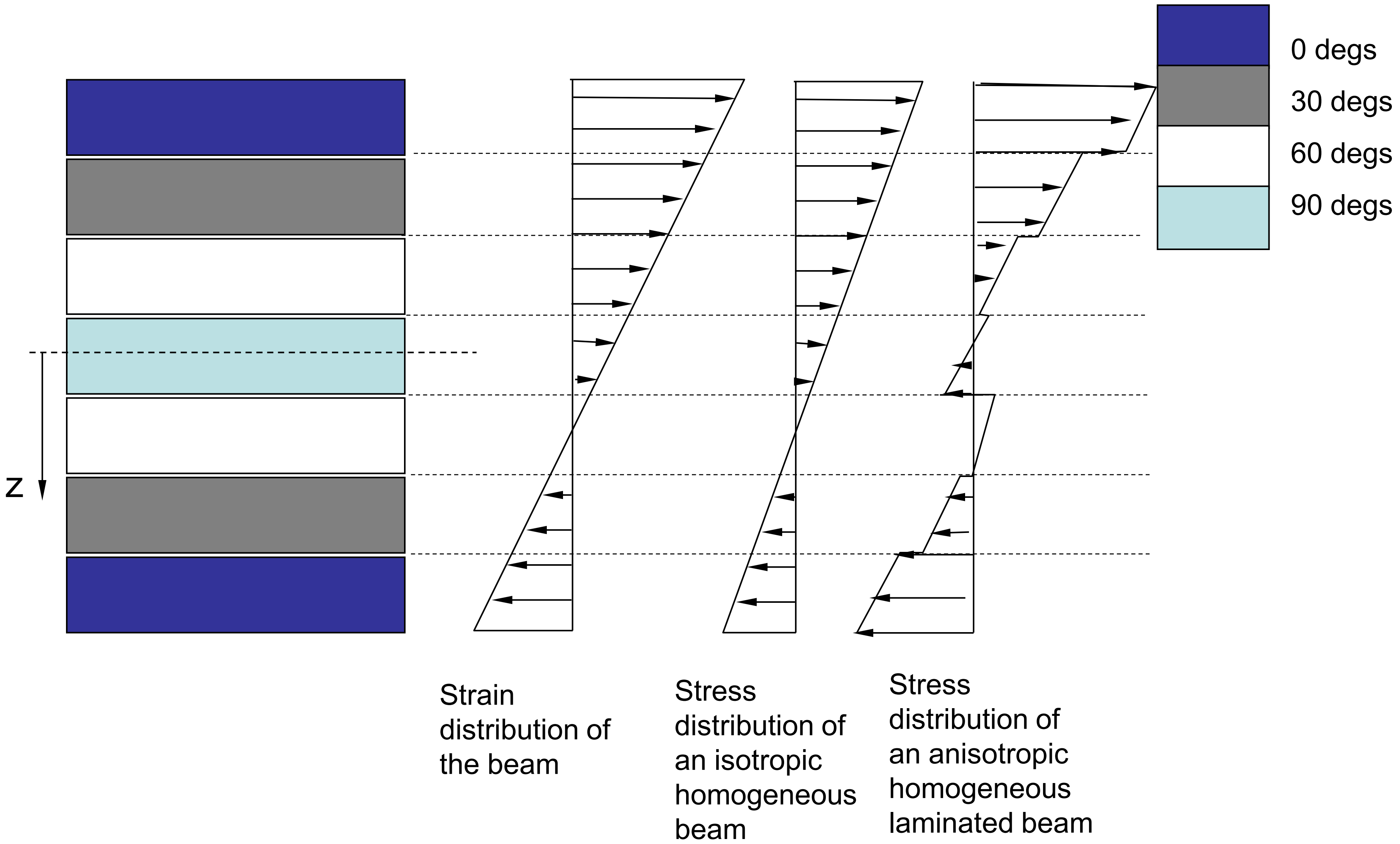
If we define as ε_0 the mid-plane strain and k the curvature of the beam then



$$\varepsilon_x = \frac{P}{EA} + z\left(\frac{M}{EI}\right) \Rightarrow \varepsilon_x = \varepsilon_0 + zk$$

This means that during simultaneous application of axial forces and bending moments the strain varies linearly through the thickness of the beam.

Stress and strain in a laminate



Stress strain relations for a laminate

- Classical Lamination Theory (CLT) is used to develop relationships between applied loads on a laminate and strains/stresses in the layers of it.

- CLT is based on the following assumptions

Assumptions of CLT

- Each lamina is orthotropic
- Each lamina is homogeneous
- A line straight and perpendicular to the middle surface remains like this during deformation, i.e., $\gamma_{xz}=0$, $\gamma_{yz}=0$
- The laminate is thin and loaded only in its plane (plane stress) which means that $\sigma_z=\tau_{xz}=\tau_{zy}=0$.
- Displacements are continuous and small
- Each lamina is elastic
- There is no delamination between successive laminae.

$$\begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{Bmatrix} = \begin{bmatrix} S_{11} & S_{12} & 0 \\ S_{21} & S_{22} & 0 \\ 0 & 0 & S_{66} \end{bmatrix} \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{Bmatrix}$$

Strains in a laminate

The displacement field for a laminate is a function of the displacements of the mid-plane. (Love-Kirchhoff theory, in fact is the 2-D extension of the Bernoulli beam theory).

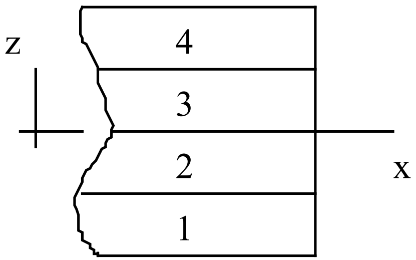
$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_s \end{Bmatrix} = \begin{Bmatrix} \varepsilon_x^o \\ \varepsilon_y^o \\ \varepsilon_s^o \end{Bmatrix} + z \begin{Bmatrix} k_x \\ k_y \\ k_s \end{Bmatrix}$$

Therefore, the strain through the thickness of a laminate is always linear…although the stress…

$$\begin{Bmatrix}\sigma_1\\ \sigma_2\\ \tau_{12}\end{Bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & 0\\ Q_{21} & Q_{22} & 0\\ 0 & 0 & Q_{66}\end{bmatrix}\begin{Bmatrix}\varepsilon_1\\ \varepsilon_2\\ \gamma_{12}\end{Bmatrix}$$

Off-axis?

$$\varepsilon = \varepsilon^{o} + zk \qquad s^{(k)} = Q^{(k)} e$$



$$\begin{Bmatrix}\sigma_x\\ \sigma_y\\ \sigma_s\end{Bmatrix}^{(k)} = \boxed{\begin{bmatrix}\bar{Q}_{xx} & \bar{Q}_{xy} & \bar{Q}_{xs}\\ \bar{Q}_{yx} & \bar{Q}_{yy} & \bar{Q}_{ys}\\ \bar{Q}_{sx} & \bar{Q}_{sy} & \bar{Q}_{ss}\end{bmatrix}^{(k)}} \left[\begin{Bmatrix}\varepsilon_x^{o}\\ \varepsilon_y^{o}\\ \varepsilon_s^{o}\end{Bmatrix} + z\begin{Bmatrix}k_x\\ k_y\\ k_s\end{Bmatrix}\right]$$

Lamina stress strain relationships

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_s \end{Bmatrix}^{(k)} = \begin{bmatrix} \overline{Q}_{xx} & \overline{Q}_{xy} & \overline{Q}_{xs} \\ \overline{Q}_{xy} & \overline{Q}_{yy} & \overline{Q}_{ys} \\ \overline{Q}_{xs} & \overline{Q}_{ys} & \overline{Q}_{ss} \end{bmatrix}^{(k)} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_s \end{Bmatrix}$$

$\overline{Q}_{ij}$ Are called the elements of the transformed reduced stiffness matrix and are given by:

$$\overline{Q}_{xx} = Q_{11}m^4 + Q_{22}n^4 + 2\left(Q_{12} + 2Q_{66}\right)m^2n^2$$

$$\overline{Q}_{xy} = \left(Q_{11} + Q_{22} - 4Q_{66}\right)m^2n^2 + Q_{12}\left(m^4 + n^4\right)$$

$$\overline{Q}_{yy} = Q_{11}n^4 + Q_{22}m^4 + 2\left(Q_{12} + 2Q_{66}\right)m^2n^2$$

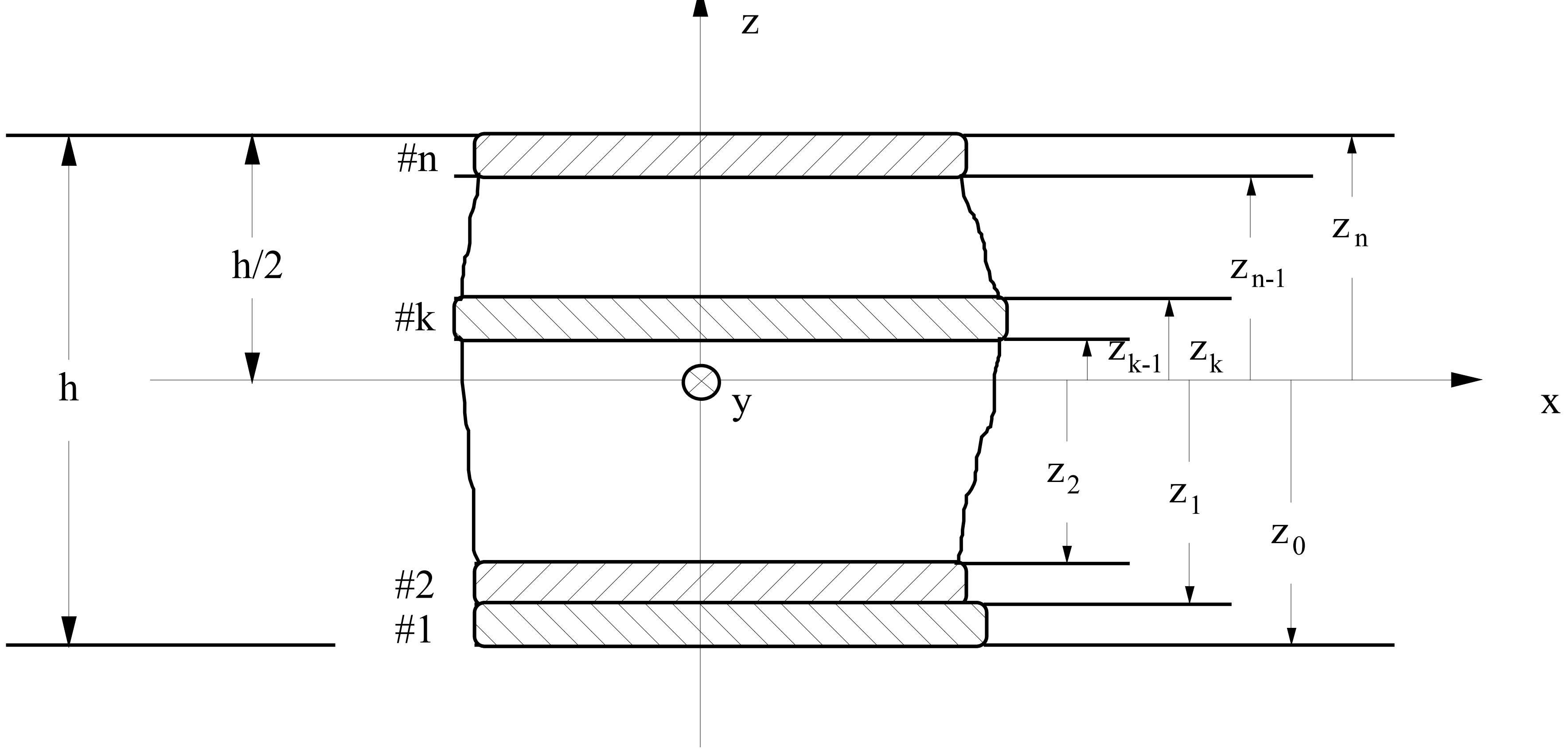
$$\overline{Q}_{xs} = \left(Q_{11} - Q_{12} - 2Q_{66}\right)m^3n - \left(Q_{22} - Q_{12} - 2Q_{66}\right)n^3m$$

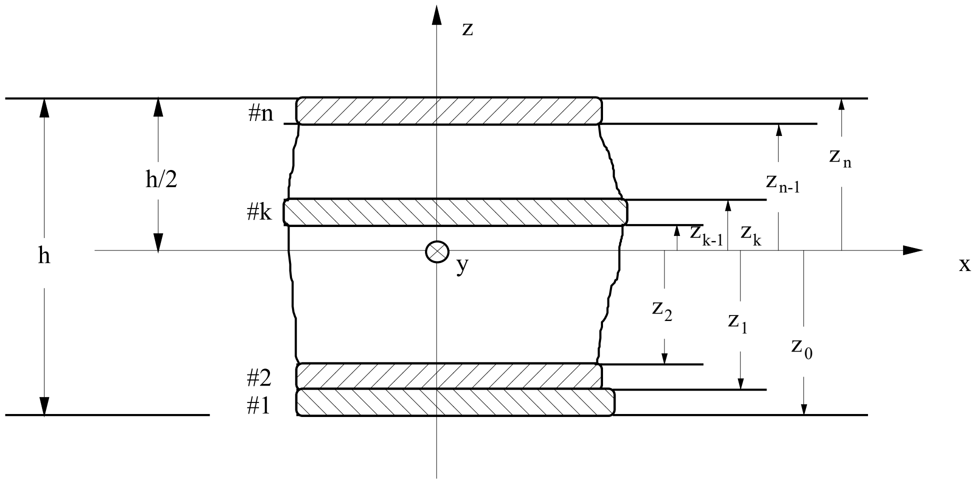
$$\overline{Q}_{ys} = \left(Q_{11} - Q_{12} - 2Q_{66}\right)mn^3 - \left(Q_{22} - Q_{12} - 2Q_{66}\right)m^3n$$

$$\overline{Q}_{ss} = \left(Q_{11} + Q_{22} - 2Q_{12} - 2Q_{66}\right)m^2n^2 + Q_{66}\left(n^4 + m^4\right)$$

Note that six elements are in the transformed stiffness matrix, however, by looking at the equations it can be seen that they are functions of the same four stiffness elements Q_{11}, Q_{12}, Q_{22}, Q_{66}

Laminate through-thickness geometry





$$\begin{Bmatrix} N_x \\ N_y \\ N_s \end{Bmatrix} = \int_{-h/2}^{h/2} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_s \end{Bmatrix} dz = \sum_{k=1}^{n} \int_{z_{k-1}}^{z_k} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_s \end{Bmatrix}^{(k)} dz$$

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_s \end{Bmatrix}^k = \begin{bmatrix} \bar{Q}_{xx} & \bar{Q}_{xy} & \bar{Q}_{xs} \\ \bar{Q}_{yx} & \bar{Q}_{yy} & \bar{Q}_{ys} \\ \bar{Q}_{sx} & \bar{Q}_{sy} & \bar{Q}_{ss} \end{bmatrix}^k \left[\begin{Bmatrix} \varepsilon_x^o \\ \varepsilon_y^o \\ \varepsilon_s^o \end{Bmatrix} + z \begin{Bmatrix} k_x \\ k_y \\ k_s \end{Bmatrix} \right]$$

$$\begin{Bmatrix} N_x \\ N_y \\ N_s \end{Bmatrix} = \sum_{k=1}^{n} \left[\int_{z_{k-1}}^{z_k} \begin{bmatrix} \bar{Q}_{xx} & \bar{Q}_{xy} & \bar{Q}_{xs} \\ \bar{Q}_{xy} & \bar{Q}_{yy} & \bar{Q}_{ys} \\ \bar{Q}_{xs} & \bar{Q}_{ys} & \bar{Q}_{ss} \end{bmatrix}^{(k)} \begin{Bmatrix} \varepsilon_\chi^o \\ \varepsilon_y^o \\ \varepsilon_s^o \end{Bmatrix} dz + \int_{z_{k-1}}^{z_k} \begin{bmatrix} \bar{Q}_{xx} & \bar{Q}_{xy} & \bar{Q}_{xs} \\ \bar{Q}_{xy} & \bar{Q}_{yy} & \bar{Q}_{ys} \\ \bar{Q}_{xs} & \bar{Q}_{ys} & \bar{Q}_{ss} \end{bmatrix}^{(k)} \begin{Bmatrix} k_x \\ k_y \\ k_s \end{Bmatrix} z dz \right]$$

The Equation

$$\begin{Bmatrix} N_x \\ N_y \\ N_s \end{Bmatrix} = \sum_{k=1}^{n} \left[\int_{z_{k-1}}^{z_k} \begin{bmatrix} \bar{Q}_{xx} & \bar{Q}_{xy} & \bar{Q}_{xs} \\ \bar{Q}_{xy} & \bar{Q}_{yy} & \bar{Q}_{ys} \\ \bar{Q}_{xs} & \bar{Q}_{ys} & \bar{Q}_{ss} \end{bmatrix}^{(\mathrm{k})} \begin{Bmatrix} \varepsilon_\chi^o \\ \varepsilon_y^o \\ \varepsilon_s^o \end{Bmatrix} dz + \int_{z_{k-1}}^{z_k} \begin{bmatrix} \bar{Q}_{xx} & \bar{Q}_{xy} & \bar{Q}_{xs} \\ \bar{Q}_{xy} & \bar{Q}_{yy} & \bar{Q}_{ys} \\ \bar{Q}_{xs} & \bar{Q}_{ys} & \bar{Q}_{ss} \end{bmatrix}^{(\mathrm{k})} \begin{Bmatrix} k_x \\ k_y \\ k_s \end{Bmatrix} zdz \right]$$

Can be re-written as

$$\begin{Bmatrix} N_x \\ N_y \\ N_s \end{Bmatrix} = \left[\sum_{k=1}^{n} \begin{bmatrix} \bar{Q}_{xx} & \bar{Q}_{xy} & \bar{Q}_{xs} \\ \bar{Q}_{xy} & \bar{Q}_{yy} & \bar{Q}_{ys} \\ \bar{Q}_{xs} & \bar{Q}_{ys} & \bar{Q}_{ss} \end{bmatrix}^{(k)} \int_{z_{k-1}}^{z_k} dz \right] \begin{Bmatrix} \varepsilon_\chi^o \\ \varepsilon_y^o \\ \varepsilon_s^o \end{Bmatrix} + \left[\sum_{k=1}^{n} \begin{bmatrix} \bar{Q}_{xx} & \bar{Q}_{xy} & \bar{Q}_{xs} \\ \bar{Q}_{xy} & \bar{Q}_{yy} & \bar{Q}_{ys} \\ \bar{Q}_{xs} & \bar{Q}_{ys} & \bar{Q}_{ss} \end{bmatrix}^{(k)} \int_{z_{k-1}}^{z_k} z\,dz \right] \begin{Bmatrix} k_x \\ k_y \\ k_s \end{Bmatrix}$$

$$\begin{Bmatrix} N_x \\ N_y \\ N_s \end{Bmatrix} = \left[\sum_{k=1}^{n} \begin{bmatrix} \bar{Q}_{xx} & \bar{Q}_{xy} & \bar{Q}_{xs} \\ \bar{Q}_{xy} & \bar{Q}_{yy} & \bar{Q}_{ys} \\ \bar{Q}_{xs} & \bar{Q}_{ys} & \bar{Q}_{ss} \end{bmatrix}^{(k)} \int_{z_{k-1}}^{z_k} dz \right] \begin{Bmatrix} \varepsilon_x^o \\ \varepsilon_y^o \\ \varepsilon_s^o \end{Bmatrix} + \left[\sum_{k=1}^{n} \begin{bmatrix} \bar{Q}_{xx} & \bar{Q}_{xy} & \bar{Q}_{xs} \\ \bar{Q}_{xy} & \bar{Q}_{yy} & \bar{Q}_{ys} \\ \bar{Q}_{xs} & \bar{Q}_{ys} & \bar{Q}_{ss} \end{bmatrix}^{(k)} \int_{z_{k-1}}^{z_k} z\, dz \right] \begin{Bmatrix} k_x \\ k_y \\ k_s \end{Bmatrix}$$

$$A_{ij} = \sum_{k=1}^{n} \bar{Q}_{ij}^{(k)} (z_k - z_{k-1})$$

$$B_{ij} = \frac{1}{2} \sum_{k=1}^{n} \bar{Q}_{ij}^{(k)} (z_k^2 - z_{k-1}^2) \,, \quad \text{i,j} = \text{x,y,s.}$$

$$\begin{Bmatrix} N_x \\ N_y \\ N_s \end{Bmatrix} = \begin{bmatrix} A_{xx} & A_{xy} & A_{xs} \\ A_{xy} & A_{yy} & A_{ys} \\ A_{xs} & A_{ys} & A_{ss} \end{bmatrix} \begin{Bmatrix} \varepsilon_x^o \\ \varepsilon_y^o \\ \varepsilon_s^o \end{Bmatrix} + \begin{bmatrix} B_{xx} & B_{xy} & B_{xs} \\ B_{xy} & B_{yy} & B_{ys} \\ B_{xs} & B_{ys} & B_{ss} \end{bmatrix} \begin{Bmatrix} k_x \\ k_y \\ k_s \end{Bmatrix}$$

$$\begin{Bmatrix} N_x \\ N_y \\ N_s \end{Bmatrix} = \begin{bmatrix} A_{xx} & A_{xy} & A_{xs} \\ A_{xy} & A_{yy} & A_{ys} \\ A_{xs} & A_{ys} & A_{ss} \end{bmatrix} \begin{Bmatrix} \varepsilon_x^o \\ \varepsilon_y^o \\ \varepsilon_s^o \end{Bmatrix} + \begin{bmatrix} B_{xx} & B_{xy} & B_{xs} \\ B_{xy} & B_{yy} & B_{ys} \\ B_{xs} & B_{ys} & B_{ss} \end{bmatrix} \begin{Bmatrix} k_x \\ k_y \\ k_s \end{Bmatrix}$$

$$A_{ij} = \sum_{k=1}^{n} \bar{Q}_{ij}^{(k)} (z_k - z_{k-1})$$

$$B_{ij} = \frac{1}{2} \sum_{k=1}^{n} \bar{Q}_{ij}^{(k)} (z_k^2 - z_{k-1}^2) \text{ , } \mathrm{i,j = x,y,s.}$$

Similarly, for the moment resultants:

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_s \end{Bmatrix}^k = \begin{bmatrix} \bar{Q}_{xx} & \bar{Q}_{xy} & \bar{Q}_{xs} \\ \bar{Q}_{yx} & \bar{Q}_{yy} & \bar{Q}_{ys} \\ \bar{Q}_{sx} & \bar{Q}_{sy} & \bar{Q}_{ss} \end{bmatrix}^k \left[\begin{Bmatrix} \varepsilon_x^o \\ \varepsilon_y^o \\ \varepsilon_s^o \end{Bmatrix} + z \begin{Bmatrix} k_x \\ k_y \\ k_s \end{Bmatrix} \right]$$

$$M = \sum_{k=1}^{n} \int_{z_{k-1}}^{z_k} \sigma^{(k)} z dz = \sum_{k=1}^{n} \left[\int_{z_{k-1}}^{z_k} \bar{Q}^{(k)} \varepsilon^o z \, dz + \int_{z_{k-1}}^{z_k} \bar{Q}^{(k)} k z^2 \, dz \right]$$

$$\begin{Bmatrix} M_x \\ M_y \\ M_s \end{Bmatrix} = \begin{bmatrix} B_{xx} & B_{xy} & B_{xs} \\ B_{xy} & B_{yy} & B_{ys} \\ B_{xs} & B_{ys} & B_{ss} \end{bmatrix} \begin{Bmatrix} \varepsilon_x^o \\ \varepsilon_y^o \\ \varepsilon_s^o \end{Bmatrix} + \begin{bmatrix} D_{xx} & D_{xy} & D_{xs} \\ D_{xy} & D_{yy} & D_{ys} \\ D_{xs} & D_{ys} & D_{ss} \end{bmatrix} \begin{Bmatrix} k_x \\ k_y \\ k_s \end{Bmatrix}$$

$$D_{ij} = \frac{1}{3} \sum_{k=1}^{n} \bar{Q}_{ij}^{(k)} (z_k^3 - z_{k-1}^3) \ , \qquad \text{i,j} = \text{x,y,s.}$$

Finally:

$$
\begin{Bmatrix} N_x \\ N_y \\ N_s \\ M_x \\ M_y \\ M_s \end{Bmatrix} = \left[\begin{array}{ccc|ccc} A_{xx} & A_{xy} & A_{xs} & B_{xx} & B_{xy} & B_{xs} \\ A_{xy} & A_{yy} & A_{ys} & B_{xy} & B_{yy} & B_{ys} \\ A_{xs} & A_{ys} & A_{ss} & B_{xs} & B_{ys} & B_{ss} \\ \hline B_{xx} & B_{xy} & B_{xs} & D_{xx} & D_{xy} & D_{xs} \\ B_{xy} & B_{yy} & B_{ys} & D_{xy} & D_{yy} & D_{ys} \\ B_{xs} & B_{ys} & B_{ss} & D_{xs} & D_{ys} & D_{ss} \end{array} \right] \begin{Bmatrix} \varepsilon_x^o \\ \varepsilon_y^o \\ \varepsilon_s^o \\ k_x \\ k_y \\ k_s \end{Bmatrix}
$$

Constitutive equations for a general laminate

Extensional

Coupling

$$
\boxed{\begin{Bmatrix} N \\ M \end{Bmatrix} = \begin{bmatrix} A & B \\ B & D \end{bmatrix} \begin{Bmatrix} \varepsilon^o \\ k \end{Bmatrix}}
$$

Bending stiffness or flexural

Coupling in the laminates

$$
\begin{Bmatrix} N_x \\ N_y \\ N_s \\ M_x \\ M_y \\ M_s \end{Bmatrix}
=
\begin{bmatrix}
A_{xx} & A_{xy} & A_{xs} & & & \\
A_{xy} & A_{yy} & A_{ys} & & & \\
A_{xs} & A_{ys} & A_{ss} & & & \\
 & & & D_{xx} & D_{xy} & D_{xs} \\
 & & & D_{xy} & D_{yy} & D_{ys} \\
 & & & D_{xs} & D_{ys} & D_{ss}
\end{bmatrix}
\begin{Bmatrix} \varepsilon_x^o \\ \varepsilon_y^o \\ \varepsilon_s^o \\ k_x \\ k_y \\ k_s \end{Bmatrix}
$$

$$N_x = A_{xx}\varepsilon_x^o + A_{xy}\varepsilon_y^o + A_{xs}\varepsilon_s^o + B_{xx}k_x + B_{xy}k_y + B_{xs}k_s$$

$$M_x = B_{xx}\varepsilon_x^o + B_{xy}\varepsilon_y^o + B_{xs}\varepsilon_s^o + D_{xx}k_x + D_{xy}k_y + D_{xs}k_s$$

A_{ij}: extensional stiffness matrix

B_{ij}: coupling stiffness matrix

D_{ij}: flexural stiffness matrix

Coupling in the laminates

$$\begin{Bmatrix} N_x \\ N_y \\ N_s \\ M_x \\ M_y \\ M_s \end{Bmatrix} = \begin{bmatrix} A_{xx} & A_{xy} & A_{xs} & B_{xx} & B_{xy} & B_{xs} \\ A_{xy} & A_{yy} & A_{ys} & B_{xy} & B_{yy} & B_{ys} \\ A_{xs} & A_{ys} & A_{ss} & B_{xs} & B_{ys} & B_{ss} \\ B_{xx} & B_{xy} & B_{xs} & D_{xx} & D_{xy} & D_{xs} \\ B_{xy} & B_{yy} & B_{ys} & D_{xy} & D_{yy} & D_{ys} \\ B_{xs} & B_{ys} & B_{ss} & D_{xs} & D_{ys} & D_{ss} \end{bmatrix} \begin{Bmatrix} \varepsilon_x^o \\ \varepsilon_y^o \\ \varepsilon_s^o \\ k_x \\ k_y \\ k_s \end{Bmatrix}$$

Non-homogeneity

Anisotropy

Example

$$\begin{Bmatrix} N_x \\ N_y \\ N_s \\ M_x \\ M_y \\ M_s \end{Bmatrix} = \begin{bmatrix} A_{xx} & A_{xy} & A_{xs} & B_{xx} & B_{xy} & B_{xs} \\ A_{xy} & A_{yy} & A_{ys} & B_{xy} & B_{yy} & B_{ys} \\ A_{xs} & A_{ys} & A_{ss} & B_{xs} & B_{ys} & B_{ss} \\ B_{xx} & B_{xy} & B_{xs} & D_{xx} & D_{xy} & D_{xs} \\ B_{xy} & B_{yy} & B_{ys} & D_{xy} & D_{yy} & D_{ys} \\ B_{xs} & B_{ys} & B_{ss} & D_{xs} & D_{ys} & D_{ss} \end{bmatrix} \begin{Bmatrix} \varepsilon_x^o \\ \varepsilon_y^o \\ \varepsilon_s^o \\ k_x \\ k_y \\ k_s \end{Bmatrix}$$

Non-homogeneity

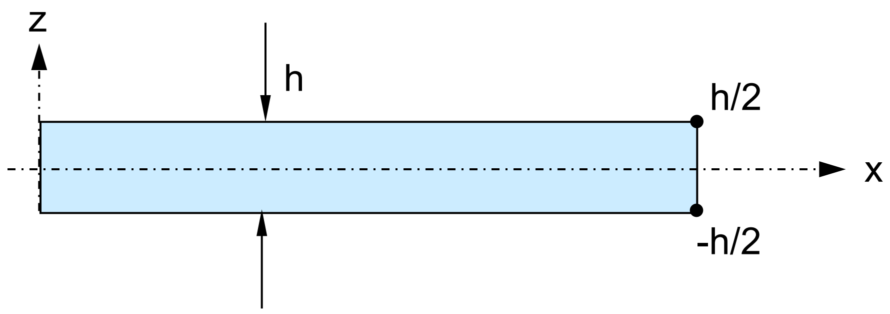
Anisotropy

$$A_{ij} = \sum_{k=1}^{n} \overline{Q}_{ij}^{(k)} (z_k - z_{k-1})$$

$$B_{ij} = \frac{1}{2} \sum_{k=1}^{n} \overline{Q}_{ij}^{(k)} (z_k^2 - z_{k-1}^2)$$

$$D_{ij} = \frac{1}{3} \sum_{k=1}^{n} \overline{Q}_{ij}^{(k)} (z_k^3 - z_{k-1}^3), \quad \text{i,j} = \text{x,y,s.}$$

Consider a laminate of one anisotropic layer



$$B_{ij} = \frac{\overline{Q}_{ij}^{k}}{2} (z_k^2 - z_{k-1}^2)$$

$$B_{ij} = \frac{\overline{Q}_{ij}^{k}}{2} \left(\left(\frac{h}{2} \right)^2 - \left(\frac{-h}{2} \right)^2 \right) = 0$$

Also, if $Q_{xs} = Q_{ys} = 0$, then $A_{xs} = A_{ys} = D_{xs} = D_{ys} = 0$

Example II

$$\begin{Bmatrix} N_x \\ N_y \\ N_s \\ M_x \\ M_y \\ M_s \end{Bmatrix} = \begin{bmatrix} A_{xx} & A_{xy} & A_{xs} & B_{xx} & B_{xy} & B_{xs} \\ A_{xy} & A_{yy} & A_{ys} & B_{xy} & B_{yy} & B_{ys} \\ A_{xs} & A_{ys} & A_{ss} & B_{xs} & B_{ys} & B_{ss} \\ B_{xx} & B_{xy} & B_{xs} & D_{xx} & D_{xy} & D_{xs} \\ B_{xy} & B_{yy} & B_{ys} & D_{xy} & D_{yy} & D_{ys} \\ B_{xs} & B_{ys} & B_{ss} & D_{xs} & D_{ys} & D_{ss} \end{bmatrix} \begin{Bmatrix} \varepsilon_x^o \\ \varepsilon_y^o \\ \varepsilon_s^o \\ k_x \\ k_y \\ k_s \end{Bmatrix}$$

Non-homogeneity

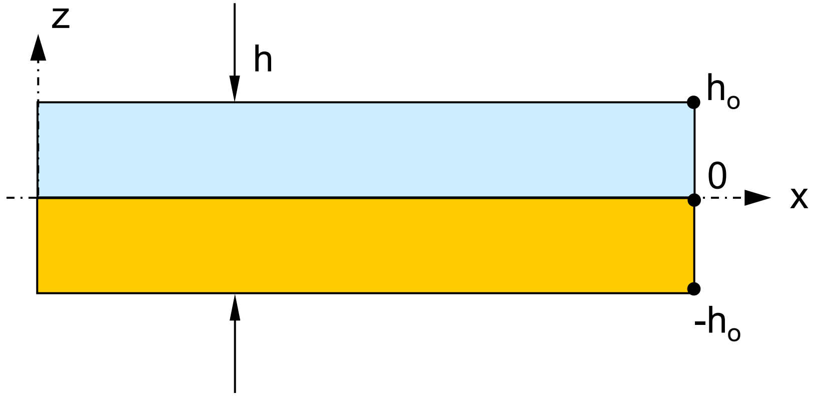
Anisotropy

$$A_{ij} = \sum_{k=1}^{n} \overline{Q}_{ij}^{(k)} (z_k - z_{k-1})$$

$$B_{ij} = \frac{1}{2} \sum_{k=1}^{n} \overline{Q}_{ij}^{(k)} (z_k^2 - z_{k-1}^2)$$

$$D_{ij} = \frac{1}{3} \sum_{k=1}^{n} \overline{Q}_{ij}^{(k)} (z_k^3 - z_{k-1}^3) \,, \quad \text{i,j} = \text{x,y,s.}$$

Consider a two layer laminate, of thickness h_o:



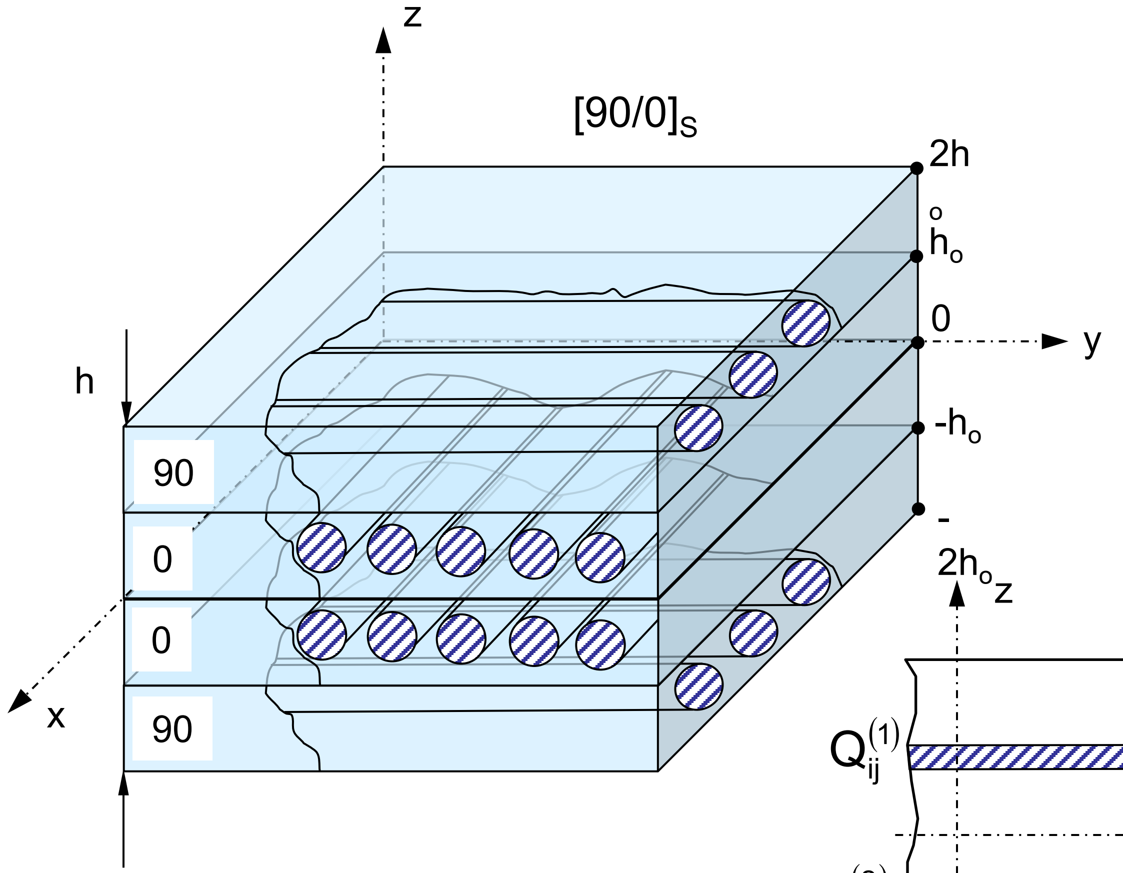
$$B_{ij} = \frac{\overline{Q}_{ij}^{(1)}}{2} (z_1^2 - z_0^2) + \frac{\overline{Q}_{ij}^{(2)}}{2} (z_2^2 - z_1^2)$$

$$B_{ij} = \frac{\overline{Q}_{ij}^{(1)}}{2} (0 - (-h_o)^2) + \frac{\overline{Q}_{ij}^{(2)}}{2} ((h_o)^2 - 0) =$$

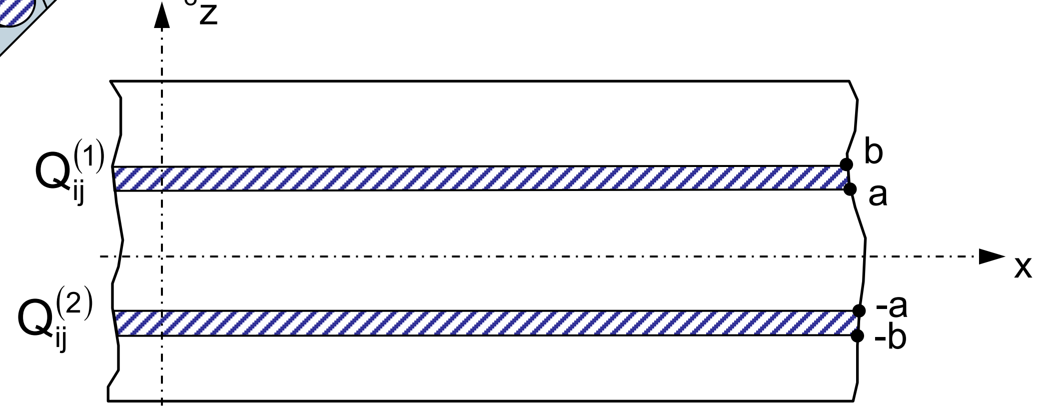
$$h_o^2 (\overline{Q}_{ij}^{(2)} - \overline{Q}_{ij}^{(1)}) \neq 0$$

Also, because $Q_{xs}=Q_{ys}=0$, obviously $A_{xs}=A_{ys}=D_{xs}=D_{ys}=0$

Symmetric laminates



$Q_{ij}\,(z)=Q_{ij}(-z)$



Calculation of coupling matrix B_{ij}:

Therefore, for symmetric laminates, the coupling matrix, **B**, is always zero.:

$$\begin{Bmatrix} N_x \\ N_y \\ N_s \end{Bmatrix} = \begin{bmatrix} A_{xx} & A_{xy} & A_{xs} \\ A_{xy} & A_{yy} & A_{ys} \\ A_{xs} & A_{ys} & A_{ss} \end{bmatrix} \begin{Bmatrix} \varepsilon_x^o \\ \varepsilon_y^o \\ \varepsilon_s^o \end{Bmatrix} \rightarrow \begin{Bmatrix} \varepsilon_x^o \\ \varepsilon_y^o \\ \varepsilon_s^o \end{Bmatrix} = \begin{bmatrix} A'_{xx} & A'_{xy} & A'_{xs} \\ A'_{xy} & A'_{yy} & A'_{ys} \\ A'_{xs} & A'_{ys} & A'_{ss} \end{bmatrix} \begin{Bmatrix} N_x \\ N_y \\ N_s \end{Bmatrix} \quad \mathbf{A'} = \mathbf{A}^{-1}$$

$$\begin{Bmatrix} M_x \\ M_y \\ M_s \end{Bmatrix} = \begin{bmatrix} D_{xx} & D_{xy} & D_{xs} \\ D_{xy} & D_{yy} & D_{ys} \\ D_{xs} & D_{ys} & D_{ss} \end{bmatrix} \begin{Bmatrix} k_x \\ k_y \\ k_s \end{Bmatrix} \rightarrow \begin{Bmatrix} k_x \\ k_y \\ k_s \end{Bmatrix} = \begin{bmatrix} D'_{xx} & D'_{xy} & D'_{xs} \\ D'_{xy} & D'_{yy} & D'_{ys} \\ D'_{xs} & D'_{ys} & D'_{ss} \end{bmatrix} \begin{Bmatrix} M_x \\ M_y \\ M_s \end{Bmatrix} \quad \mathbf{D'} = \mathbf{D}^{-1}$$

If we dividing by the thickness of the laminate, h and solve for strains:

$$\begin{Bmatrix} \bar{\sigma}_x \\ \bar{\sigma}_y \\ \bar{\sigma}_s \end{Bmatrix} = \begin{bmatrix} Q'_{xx} & Q'_{xy} & Q'_{xs} \\ Q'_{xy} & Q'_{yy} & Q'_{ys} \\ Q'_{xs} & Q'_{ys} & Q'_{ss} \end{bmatrix} \begin{Bmatrix} \varepsilon_x^o \\ \varepsilon_y^o \\ \varepsilon_s^o \end{Bmatrix}$$

where: $\bar{\sigma}_i = \dfrac{N_i}{h}, \quad Q'_{ij} = \dfrac{A_{ij}}{h}$

$$\begin{Bmatrix} \varepsilon_x^o \\ \varepsilon_y^o \\ \varepsilon_s^o \end{Bmatrix} = \begin{bmatrix} \bar{S}_{xx} & \bar{S}_{xy} & \bar{S}_{xs} \\ \bar{S}_{xy} & \bar{S}_{yy} & \bar{S}_{ys} \\ \bar{S}_{xs} & \bar{S}_{ys} & \bar{S}_{ss} \end{bmatrix} \begin{Bmatrix} \bar{\sigma}_x \\ \bar{\sigma}_y \\ \bar{\sigma}_s \end{Bmatrix}$$

$$\bar{S}_{ij} = hA_{ij}^{-1} = hA'_{ij}$$

Effective properties

Engineering constants for symmetric laminates

$$\begin{Bmatrix} \varepsilon_x^o \\ \varepsilon_y^o \\ \varepsilon_s^o \end{Bmatrix} = \begin{bmatrix} \bar{S}_{xx} & \bar{S}_{xy} & \bar{S}_{xs} \\ \bar{S}_{xy} & \bar{S}_{yy} & \bar{S}_{ys} \\ \bar{S}_{xs} & \bar{S}_{ys} & \bar{S}_{ss} \end{bmatrix} \begin{Bmatrix} \bar{\sigma}_x \\ \bar{\sigma}_y \\ \bar{\sigma}_s \end{Bmatrix}$$

$$\bar{S}_{ij} = hA_{ij}^{-1} = hA'_{ij}$$

For symmetric, balanced laminates:

$$A'_{xx} = \frac{A_{yy}}{A} \qquad A'_{xy} = -\frac{A_{xy}}{A} \qquad A = A_{xx}A_{yy} - A_{xy}^2$$

$$A'_{yy} = \frac{A_{xx}}{A} \qquad A'_{ss} = \frac{1}{A_{ss}}$$

$$E_x = \frac{A_{xx}A_{yy} - A_{xy}^2}{hA_{yy}} \qquad E_y = \frac{A_{xx}A_{yy} - A_{xy}^2}{hA_{xx}}$$

$$\nu_{xy} = \frac{A_{xy}}{A_{yy}} \qquad \nu_{yx} = \frac{A_{xy}}{A_{xx}} \qquad G_{xy} = \frac{A_{ss}}{h}$$

Bending of symmetric laminates

$$\begin{Bmatrix} M_x \\ M_y \\ M_s \end{Bmatrix} = \begin{bmatrix} \mathrm{D}_{xx} & \mathrm{D}_{xy} & \mathrm{D}_{xs} \\ \mathrm{D}_{xy} & \mathrm{D}_{yy} & \mathrm{D}_{ys} \\ \mathrm{D}_{xs} & \mathrm{D}_{ys} & \mathrm{D}_{ss} \end{bmatrix} \begin{Bmatrix} k_x \\ k_y \\ k_s \end{Bmatrix} \longrightarrow \begin{Bmatrix} k_x \\ k_y \\ k_s \end{Bmatrix} = \begin{bmatrix} \mathrm{D}'_{xx} & \mathrm{D}'_{xy} & \mathrm{D}'_{xs} \\ \mathrm{D}'_{xy} & \mathrm{D}'_{yy} & \mathrm{D}'_{ys} \\ \mathrm{D}'_{xs} & \mathrm{D}'_{ys} & \mathrm{D}'_{ss} \end{bmatrix} \begin{Bmatrix} M_x \\ M_y \\ M_s \end{Bmatrix} \quad \mathbf{D}' = \mathbf{D}^{-1}$$

$$D_{ij} = \frac{1}{3}\sum_{k=1}^{n} \overline{Q}_{ij}^{(k)} (z_k^3 - z_{k-1}^3) \,, \quad \mathrm{i,j = x,y,s.}$$

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_s \end{Bmatrix}^k = \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_s \end{Bmatrix}^k = z \begin{bmatrix} \overline{Q}_{xx} & \overline{Q}_{xy} & \overline{Q}_{xs} \\ \overline{Q}_{yx} & \overline{Q}_{yy} & \overline{Q}_{ys} \\ \overline{Q}_{sx} & \overline{Q}_{sy} & \overline{Q}_{ss} \end{bmatrix}^k \begin{Bmatrix} k_x \\ k_y \\ k_s \end{Bmatrix}$$

0

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_s \end{Bmatrix} = z \begin{Bmatrix} k_x \\ k_y \\ k_s \end{Bmatrix}$$

Flexural engineering constants for symmetric laminates

Consider a beam of thickness h and width b from orthotropic material (on-axis):

$$D_{ij} = \frac{1}{3}\sum_{k=1}^{n} \overline{Q}_{ij}^{(k)} (z_k^3 - z_{k-1}^3)\,, \quad \mathrm{i,j = x,y,s.} \longrightarrow D_{ij} = \frac{h^3}{12}\overline{Q}_{ij}$$

$$D'_{xx} = \frac{D_{yy}}{D_{xx}D_{yy} - D_{xy}^2} \longrightarrow D'_{xx} = \frac{12}{h^3}\frac{Q_{yy}}{Q_{xx}Q_{yy} - Q_{xy}^2} = \frac{12}{h^3}S_{xx} = \frac{12}{h^3 E_x}$$

$$\boxed{E_x^f = E_x} \longleftarrow E_x^f = \frac{12}{h^3 D'_{xx}}$$

$$E_y^f = \frac{12}{h^3 D'_{yy}} \qquad E_s^f = \frac{12}{h^3 D'_{ss}}$$

Application of CLT

Problem statement:

Given the applied force and moment resultants on a laminate, calculate stresses and strains at each lamina

Solution (6 steps)

STEP 1. For each ply calculate:

$$Q_{11} = \frac{E_1}{1 - \nu_{21}\nu_{12}}$$

$$Q_{12} = \frac{\nu_{12}E_2}{1 - \nu_{21}\nu_{12}}$$

$$Q_{22} = \frac{E_2}{1 - \nu_{21}\nu_{12}}$$

$$Q_{66} = G_{12}$$

$$\overline{Q}_{xx} = Q_{11}m^4 + Q_{22}n^4 + 2(Q_{12} + 2Q_{66})m^2n^2$$

$$\overline{Q}_{xy} = (Q_{11} + Q_{22} - 4Q_{66})m^2n^2 + Q_{12}(m^4 + n^4)$$

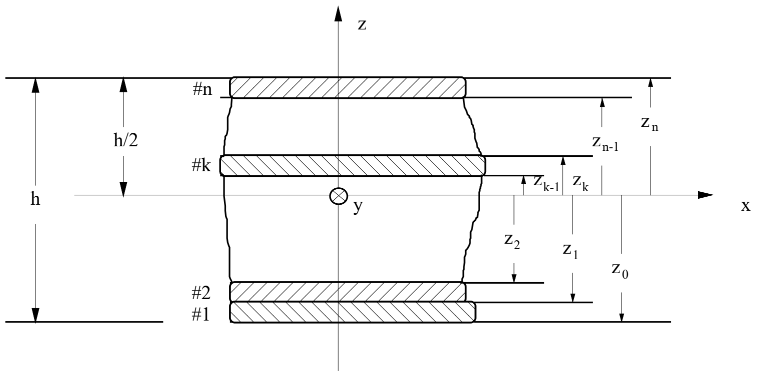
$$\overline{Q}_{yy} = Q_{11}n^4 + Q_{22}m^4 + 2(Q_{12} + 2Q_{66})m^2n^2$$

$$\overline{Q}_{xs} = (Q_{11} - Q_{12} - 2Q_{66})m^3n - (Q_{22} - Q_{12} - 2Q_{66})n^3m$$

$$\overline{Q}_{ys} = (Q_{11} - Q_{12} - 2Q_{66})mn^3 - (Q_{22} - Q_{12} - 2Q_{66})m^3n$$

$$\overline{Q}_{ss} = (Q_{11} + Q_{22} - 2Q_{12} - 2Q_{66})m^2n^2 + Q_{66}(n^4 + m^4)$$

STEP 2. For each ply calculate top and bottom layer coordinates:



$$h = \sum_{k=1}^{n} t_k$$

$$Ply\ 1:\ z_0 = -\frac{z}{2}\ \left(top\ surface\right),$$

$$z_1 = -\frac{z}{2} + t_1\ \left(bottom\ surface\right)$$

$$Ply\ k:\ z_{k-1} = -\frac{z}{2} + \sum_{m=1}^{k-1} t_m\ \left(top\ surface\right)$$

$$z_k = -\frac{z}{2} + + \sum_{m=1}^{k} t_m\ \left(bottom\ surface\right)$$

STEP 3. Calculate elements of the Extensional [A], Coupling [B], and Bending stiffness [D] matrices for the <u>entire</u> laminate

$$A_{ij} = \sum_{k=1}^{n} \overline{Q}_{ij}^{(k)} (z_k - z_{k-1})$$

$$B_{ij} = \frac{1}{2} \sum_{k=1}^{n} \overline{Q}_{ij}^{(k)} (z_k^2 - z_{k-1}^2)$$

$$D_{ij} = \frac{1}{3} \sum_{k=1}^{n} \overline{Q}_{ij}^{(k)} (z_k^3 - z_{k-1}^3)\ , \quad \text{i,j} = 1,2,6.$$

STEP 4. Solve 6 equations and calculate LAMINATES mid-plain strains and curvatures

$$\begin{Bmatrix} N_x \\ N_y \\ N_s \\ M_x \\ M_y \\ M_s \end{Bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{16} & B_{11} & B_{12} & B_{16} \\ A_{12} & A_{22} & A_{26} & B_{12} & B_{22} & B_{26} \\ A_{16} & A_{26} & A_{66} & B_{16} & B_{26} & B_{66} \\ B_{11} & B_{12} & B_{16} & D_{11} & D_{12} & D_{16} \\ B_{12} & B_{22} & B_{26} & D_{12} & D_{22} & D_{26} \\ B_{16} & B_{26} & B_{66} & D_{16} & D_{26} & D_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \varepsilon_s^0 \\ k_x \\ k_y \\ k_s \end{Bmatrix}$$

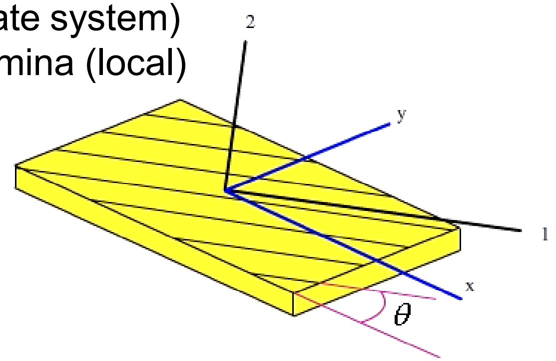
STEP 5. With the location of each layer known, calculate global strains in each ply:

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_s \end{Bmatrix} = \begin{Bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \varepsilon_s^0 \end{Bmatrix} + z \begin{Bmatrix} k_x \\ k_y \\ k_s \end{Bmatrix}$$

STEP 6. Use stress-strain equations to calculate global stresses for each LAMINA.

$$\begin{Bmatrix}\sigma_x\\ \sigma_y\\ \sigma_s\end{Bmatrix}^{(k)} = \begin{bmatrix}\bar{Q}_{xx} & \bar{Q}_{xy} & \bar{Q}_{xs}\\ \bar{Q}_{xy} & \bar{Q}_{yy} & \bar{Q}_{ys}\\ \bar{Q}_{xs} & \bar{Q}_{ys} & \bar{Q}_{ss}\end{bmatrix}^{(k)} \begin{Bmatrix}\varepsilon_x\\ \varepsilon_y\\ \varepsilon_s\end{Bmatrix}$$

x, y is the global (laminate coordinate system)
1, 2 are the principal axes of the lamina (local)



Local strains or stresses (at the fibers coordination system) can then easily be calculated using the transformation equations:

$$\begin{Bmatrix}\sigma_1\\ \sigma_2\\ \sigma_6\end{Bmatrix} = \begin{bmatrix}m^2 & n^2 & 2mn\\ n^2 & m^2 & -2mn\\ -mn & mn & m^2-n^2\end{bmatrix} \begin{Bmatrix}\sigma_x\\ \sigma_y\\ \sigma_s\end{Bmatrix}$$

$$m = \cos(\theta)$$
$$n = \sin(\theta)$$

OR $$\begin{Bmatrix}\varepsilon_1\\ \varepsilon_2\\ \varepsilon_6\end{Bmatrix} = \begin{bmatrix}m^2 & n^2 & mn\\ n^2 & m^2 & -mn\\ -2mn & 2mn & m^2-n^2\end{bmatrix} \begin{Bmatrix}\varepsilon_x\\ \varepsilon_y\\ \varepsilon_s\end{Bmatrix}$$

Transformation of stresses

$$\begin{bmatrix}\sigma_1\\ \sigma_2\\ \sigma_6\end{bmatrix} = \begin{bmatrix} m^2 & n^2 & 2mn \\ n^2 & m^2 & -2mn \\ -mn & mn & m^2-n^2 \end{bmatrix} \begin{bmatrix}\sigma_x\\ \sigma_y\\ \sigma_s\end{bmatrix}$$

$$m = \cos(\theta)$$

$$n = \sin(\theta)$$

Stress transformation is independent of the material system. It is only related to the angle (counted anticlockwise) between the two coordinate systems (in this case from x to 1).

$$[T] = \begin{bmatrix} m^2 & n^2 & 2mn \\ n^2 & m^2 & -2mn \\ -mn & mn & m^2-n^2 \end{bmatrix} \Longrightarrow [T]^{-1} = \begin{bmatrix} m^2 & n^2 & -2mn \\ n^2 & m^2 & 2mn \\ mn & -mn & m^2-n^2 \end{bmatrix}$$

$$\begin{bmatrix}\sigma_x\\ \sigma_y\\ \sigma_s\end{bmatrix} = \begin{bmatrix} m^2 & n^2 & -2mn \\ n^2 & m^2 & 2mn \\ mn & -mn & m^2-n^2 \end{bmatrix} \begin{bmatrix}\sigma_1\\ \sigma_2\\ \sigma_6\end{bmatrix}$$

Transformation of strains

$$\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_6 \end{bmatrix} = \begin{bmatrix} m^2 & n^2 & mn \\ n^2 & m^2 & -mn \\ -2mn & 2mn & m^2 - n^2 \end{bmatrix} \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_s \end{bmatrix}$$

$$m = \cos(\theta)$$

$$n = \sin(\theta)$$

$$[T] = \begin{bmatrix} m^2 & n^2 & mn \\ n^2 & m^2 & -mn \\ -2mn & 2mn & m^2 - n^2 \end{bmatrix} \Rightarrow [T]^{-1} = \begin{bmatrix} m^2 & n^2 & -mn \\ n^2 & m^2 & mn \\ 2mn & -2mn & m^2 - n^2 \end{bmatrix}$$

$$\begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_s \end{bmatrix} = \begin{bmatrix} m^2 & n^2 & -mn \\ n^2 & m^2 & mn \\ 2mn & -2mn & m^2 - n^2 \end{bmatrix} \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_6 \end{bmatrix}$$

Strain transformation is independent of the material system. It is only related to the angle (counted anticlockwise) between the two coordinate systems.

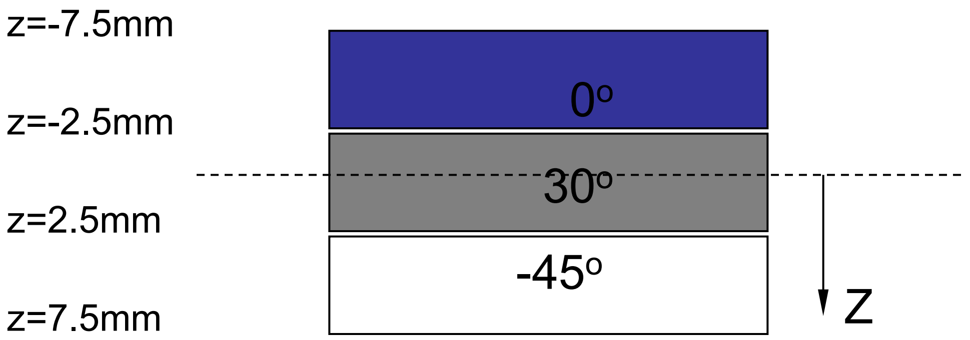
Example

Consider a Graphite/Epoxy laminate with stacking sequence $[0/30/-45]_T$. Each layer has a thickness of 5mm. The laminate is subjected to a load of $N_x = N_y = 1000$N/m.

Material properties: E_1=181 GPa, E_2=10.3 GPa, v_{12}=0.28 and G_{12}=7.17 GPa.

Calculate

- Mid-plane strains and curvatures for the laminate
- Global and local stresses on the top surface of 30^o ply



z_o=-0.0075m

z_1=-0.0025m

z_2=0.0025m

z_3=0.0075m

STEP 1. Calculation of reduced stiffness matrix for each ply:

$$Q_{11} = \frac{E_1}{1-\nu_{21}\nu_{12}} = \frac{181\cdot 10^9\,Pa}{1-0.016\cdot 0.28} = 181.8\cdot 10^9\,Pa$$

$$Q_{12} = \frac{\nu_{12}E_2}{1-\nu_{21}\nu_{12}} = \frac{0.28\cdot 10.3\cdot 10^9\,Pa}{1-0.016\cdot 0.28} = 2.897\cdot 10^9\,Pa$$

$$Q_{22} = \frac{E_2}{1-\nu_{21}\nu_{12}} = \frac{10.3\cdot 10^9\,Pa}{1-0.016\cdot 0.28} = 10.35\cdot 10^9\,Pa$$

$$Q_{66} = G_{12} = 7.17\cdot 10^9\,Pa$$

$$[\bar{Q}]_0 = \begin{bmatrix} 181.80 & 2.90 & 0 \\ 2.90 & 10.35 & 0 \\ 0 & 0 & 7.17 \end{bmatrix} \cdot 10^9 Pa$$

$$[\bar{Q}]_{-45} = \begin{bmatrix} 56.66 & 42.32 & -42.87 \\ 42.32 & 56.66 & -42.87 \\ -42.87 & -42.87 & 46.59 \end{bmatrix} \cdot 10^9\,Pa$$

$$[\bar{Q}]_{30} = \begin{bmatrix} 109.40 & 32.46 & 54.19 \\ 32.46 & 23.65 & 20.05 \\ 54.19 & 20.05 & 36.74 \end{bmatrix} \cdot 10^9\,Pa$$

STEP 2. see previous slide:

STEP 3. Calculation of extensional, coupling and bending stiffness matrices for the LAMINATE:

$$A_{ij} = \sum_{k=1}^{n} \overline{Q}_{ij}^{(k)} (z_k - z_{k-1}), \text{ i,j} = 1,2,6.$$

$$[A] = \begin{bmatrix} 181.80 & 2.90 & 0 \\ 2.90 & 10.35 & 0 \\ 0 & 0 & 7.17 \end{bmatrix} \cdot 10^9\, Pa \cdot \left[(-0.0025) - (-0.0075)\right] m$$

$$+ \begin{bmatrix} 109.40 & 32.46 & 54.19 \\ 32.46 & 23.65 & 20.05 \\ 54.19 & 20.05 & 36.74 \end{bmatrix} \cdot 10^9\, Pa \cdot \left[0.0025 - (-0.0025)\right] m$$

$$+ \begin{bmatrix} 56.66 & 42.32 & -42.87 \\ 42.32 & 56.66 & -42.87 \\ -42.87 & -42.87 & 46.59 \end{bmatrix} \cdot 10^9\, Pa \cdot \left[0.0075 - 0.0025\right] m$$

$$[A] = \begin{bmatrix} 1.74 \cdot 10^9 & 3.88 \cdot 10^8 & 5.66 \cdot 10^7 \\ 3.88 \cdot 10^8 & 4.53 \cdot 10^8 & -1.14 \cdot 10^8 \\ 5.66 \cdot 10^7 & -1.14 \cdot 10^8 & 4.52 \cdot 10^8 \end{bmatrix} Pa\, m$$

STEP 3. Calculation of extensional, coupling and bending stiffness matrices for the LAMINATE:

$$[B] = \begin{bmatrix} -3.13\cdot 10^6 & 9.86\cdot 10^5 & -1.07\cdot 10^6 \\ 9.86\cdot 10^5 & 1.16\cdot 10^6 & -1.07\cdot 10^6 \\ -1.07\cdot 10^6 & -1.07\cdot 10^6 & 9.86\cdot 10^5 \end{bmatrix} Pa\ m^2$$

$$[D] = \begin{bmatrix} 3.34\cdot 10^4 & 6.46\cdot 10^3 & -5.24\cdot 10^3 \\ 6.46\cdot 10^3 & 9.32\cdot 10^3 & -5.60\cdot 10^3 \\ -5.24\cdot 10^3 & -5.60\cdot 10^3 & 7.66\cdot 10^3 \end{bmatrix} Pa\ m^3$$

STEP 4. Solve 6 equations for LAMINATE'S strains and curvatures

$$\begin{Bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \\ \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{Bmatrix} = \left[\begin{array}{ccc|ccc} 1.30E-09 & -9.84E-10 & -6.04E-10 & 1.78E-07 & -1.10E-07 & 1.63E-07 \\ -9.84E-10 & 4.48E-09 & -1.56E-10 & -1.48E-07 & -2.19E-07 & 2.48E-07 \\ -6.04E-10 & -1.56E-10 & 3.71E-09 & -5.49E-08 & 3.12E-07 & -3.93E-07 \\ \hline 1.78E-07 & -1.48E-07 & -5.49E-08 & 5.97E-05 & -3.00E-05 & 3.01E-05 \\ -1.10E-07 & -2.19E-07 & 3.12E-07 & -3.00E-05 & 2.47E-04 & 7.38E-05 \\ 1.63E-07 & 2.48E-07 & -3.93E-07 & 3.01E-05 & 7.38E-05 & 3.13E-04 \end{array}\right] \begin{Bmatrix} 1000 \\ 1000 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$\begin{Bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \\ \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{Bmatrix} = \begin{Bmatrix} 3.13E-07 \\ 3.49E-06 \\ -7.60E-07 \\ 2.98E-05 \\ -3.29E-04 \\ 4.10E-04 \end{Bmatrix} \begin{matrix} \\ \text{m/m} \\ \\ \\ \text{1/m} \\ \\ \end{matrix}$$

Note that four parts of the compliance matrix are not the inverted A, B and D matrices!!!

STEP 5. Calculation of global strains at the top surface of 30° ply:

z distance of the top surface of 30° ply is -0.0025m

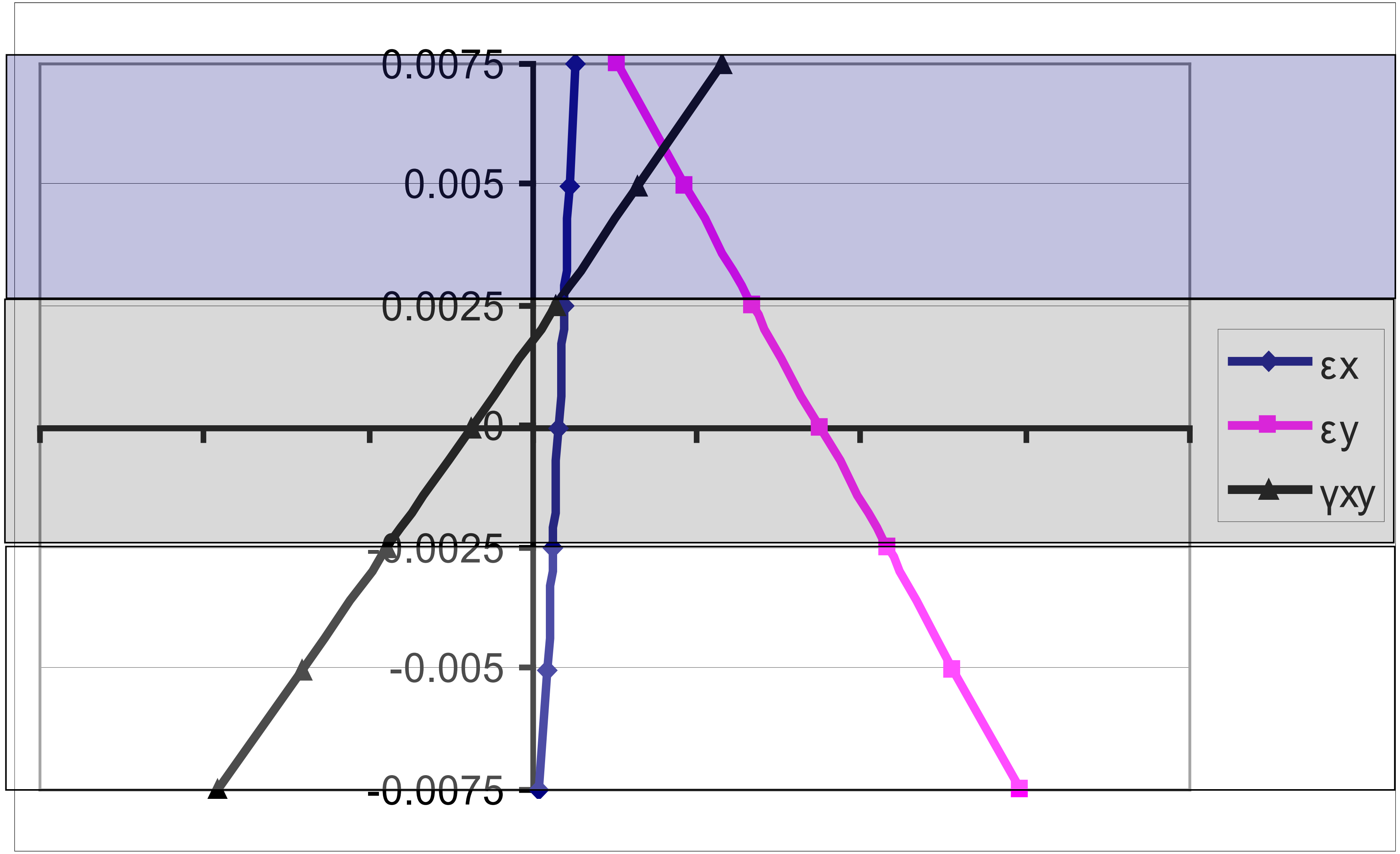
$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_s \end{Bmatrix}_{30^o,top} = \begin{bmatrix} 3.13\text{E-}07 \\ 3.49\text{E-}06 \\ -7.60\text{E-}07 \end{bmatrix} + (-0.0025)\begin{Bmatrix} 2.98\text{E-}05 \\ -3.29\text{E-}04 \\ 4.10\text{E-}04 \end{Bmatrix} = \begin{Bmatrix} 2.38\text{E-}07 \\ 4.31\text{E-}06 \\ -1.79\text{E-}06 \end{Bmatrix} m/m$$

STEP 6. Using stress strain relationship for 30° ply the global stesses can be calculated as:

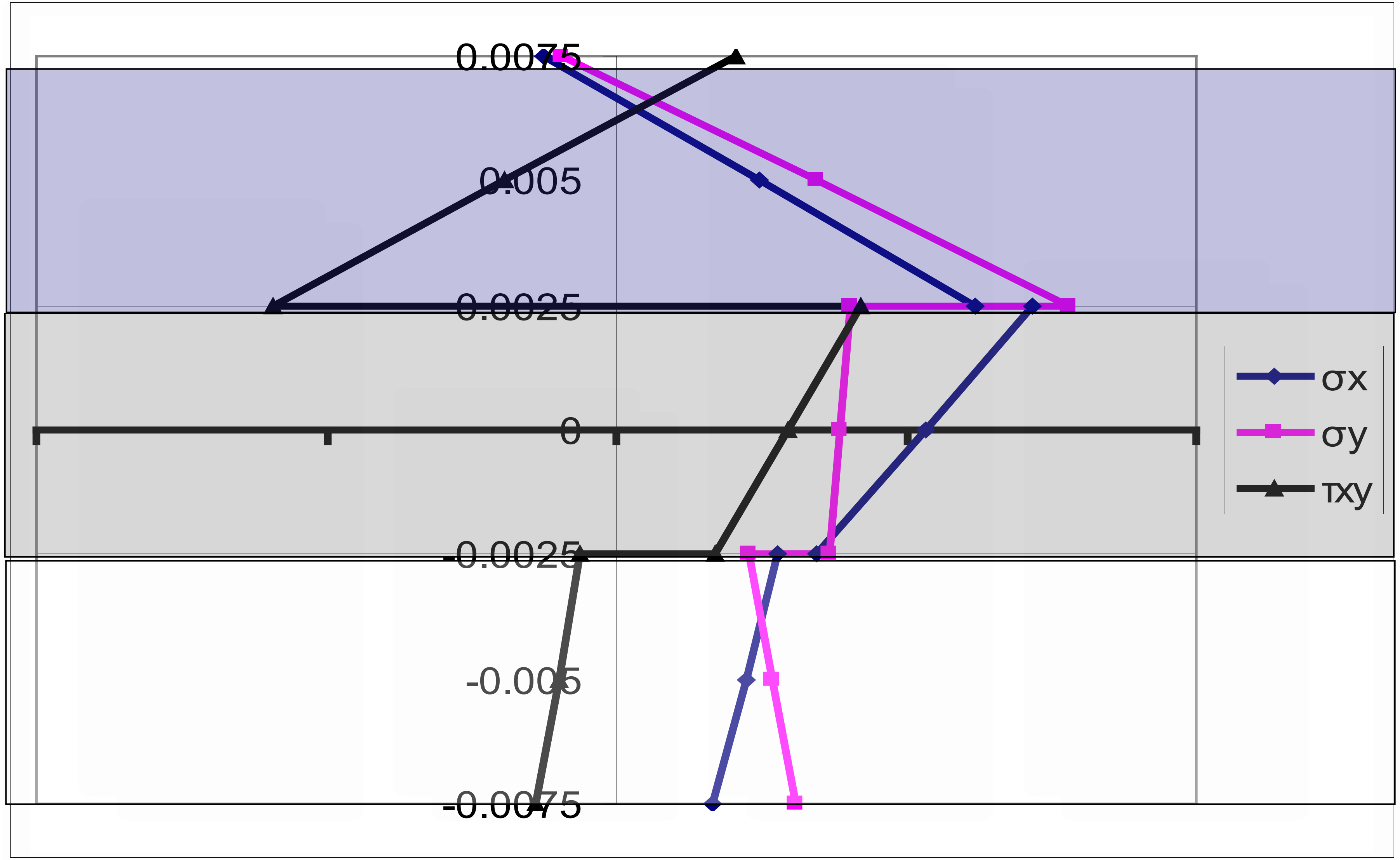
$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_s \end{Bmatrix}_{30^o,top} = \begin{bmatrix} 109.40 & 32.46 & 54.19 \\ 32.46 & 23.65 & 20.05 \\ 54.19 & 20.05 & 36.74 \end{bmatrix} \cdot 10^9 \cdot \begin{Bmatrix} 2.38\text{E-}07 \\ 4.31\text{E-}06 \\ -1.79\text{E-}06 \end{Bmatrix} Pa = \begin{Bmatrix} 6.93E+04 \\ 7.39E+04 \\ 3.38E+04 \end{Bmatrix} Pa$$

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_6 \end{Bmatrix}_{30^o,top} = \begin{bmatrix} 0.75 & 0.25 & 0.87 \\ 0.25 & 0.75 & -0.87 \\ -0.43 & 0.43 & 0.50 \end{bmatrix} \cdot \begin{Bmatrix} 6.93E+04 \\ 7.39E+04 \\ 3.38E+04 \end{Bmatrix} Pa = \begin{Bmatrix} 9.97E+04 \\ 4.35E+04 \\ 1.89E+04 \end{Bmatrix} Pa$$

		Global strains			Global stresses		
	z	εx	εy	γxy	σx	σy	τxy
0 top	-0.0075	8.94E-08	5.96E-06	-3.84E-06	3.35E+04	6.19E+04	-2.75E+04
middle	-0.005	1.64E-07	5.13E-06	-2.81E-06	4.46E+04	5.36E+04	-2.02E+04
bottom	-0.0025	2.38E-07	4.31E-06	-1.79E-06	5.58E+04	4.53E+04	-1.28E+04
30 top	-0.0025	2.38E-07	4.31E-06	-1.79E-06	6.93E+04	7.39E+04	3.38E+04
middle	0	3.12E-07	3.49E-06	-7.60E-07	1.06E+05	7.75E+04	5.90E+04
bottom	0.0025	3.87E-07	2.67E-06	2.66E-07	1.43E+05	8.10E+04	8.43E+04
-45 top	0.0025	3.87E-07	2.67E-06	2.66E-07	1.24E+05	1.56E+05	-1.19E+05
middle	0.005	4.61E-07	1.85E-06	1.29E-06	4.90E+04	6.89E+04	-3.89E+04
bottom	0.0075	5.35E-07	1.03E-06	2.32E-06	-2.55E+04	-1.84E+04	4.09E+04

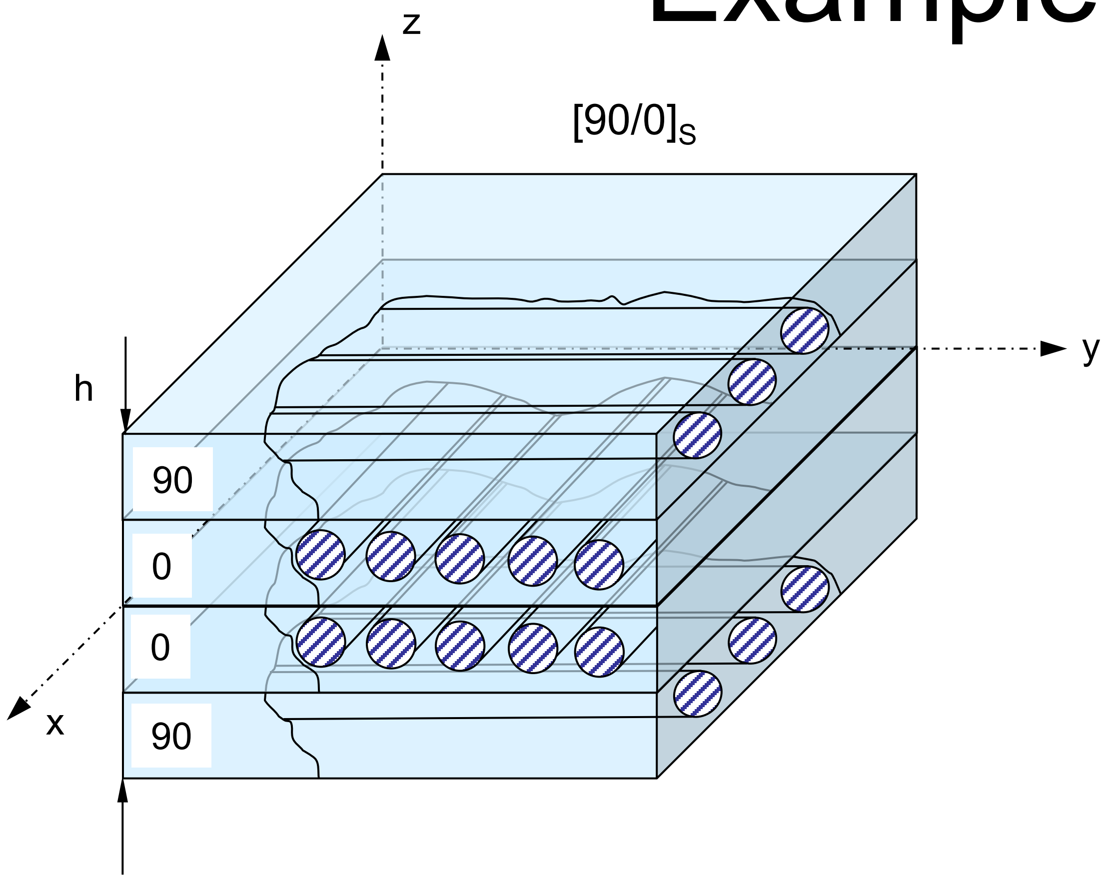


Global strains through the thickness of the laminate



Global stresses through the thickness of the laminate

Example



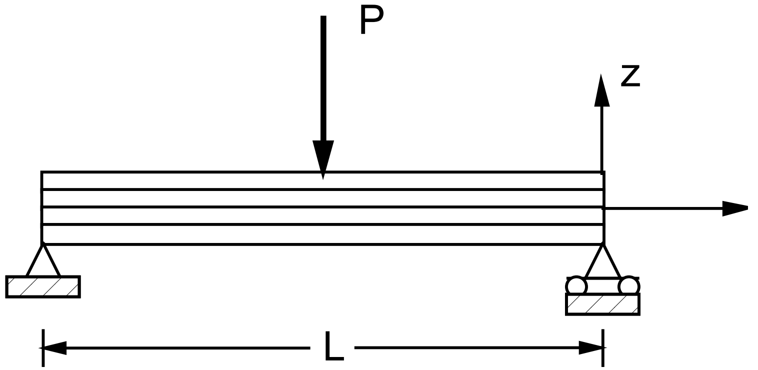
$$\boxed{E_x = E_y}$$

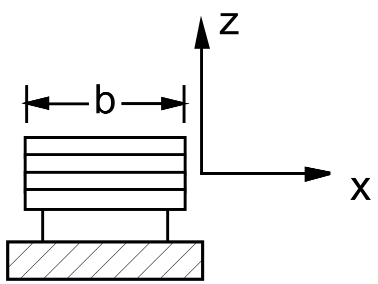
$$\boxed{E_x^f < E_y^f}$$

$$E_x = \frac{A_{xx}A_{yy} - A_{xy}^2}{hA_{yy}} \qquad E_y = \frac{A_{xx}A_{yy} - A_{xy}^2}{hA_{xx}}$$

Example: Define the stress strain field for the simply supported laminate under the three point bending conditions

Cross-ply, $[0_4/90_4]_S$, T300/5208 (properties in lecture III)





$$D'_{xx} = 9.36\cdot 10^{-3}(mN)^{-1}$$
$$D'_{xy} = -8.52\cdot 10^{-4}$$

h_o = 125 10^{-6} m , P = 100 N , L = 0.1 m , b = 0.01 m

M = PL/4 $\rightarrow$ $$\begin{Bmatrix} M_x \\ M_y \\ M_s \end{Bmatrix} = \begin{Bmatrix} M/b \\ 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} -PL/4b \\ 0 \\ 0 \end{Bmatrix}$$

$$\begin{Bmatrix} k_x \\ k_y \\ k_s \end{Bmatrix} = \begin{bmatrix} D'_{xx} & D'_{xy} & \cancel{D'_{xs}} \\ D'_{xy} & D'_{yy} & \cancel{D'_{ys}} \\ \cancel{D'_{xs}} & \cancel{D'_{ys}} & D'_{ss} \end{bmatrix} \begin{Bmatrix} M_x \\ \cancel{M_y} \\ \cancel{M_s} \end{Bmatrix} \rightarrow \begin{matrix} k_x = D'_{xx} M_x \\ k_y = D'_{yx} M_x \\ k_s = 0 \end{matrix} \rightarrow \begin{matrix} k_x = -2.34 \text{ m}^{-1} \\ k_y = 0.212 \text{ m}^{-1} \\ k_s = 0 \end{matrix}$$

And the maximum deflection at L/2:

Knowing the strain variation through the thickness:
$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_s \end{Bmatrix} = z \begin{Bmatrix} k_x \\ k_y \\ k_s \end{Bmatrix}$$

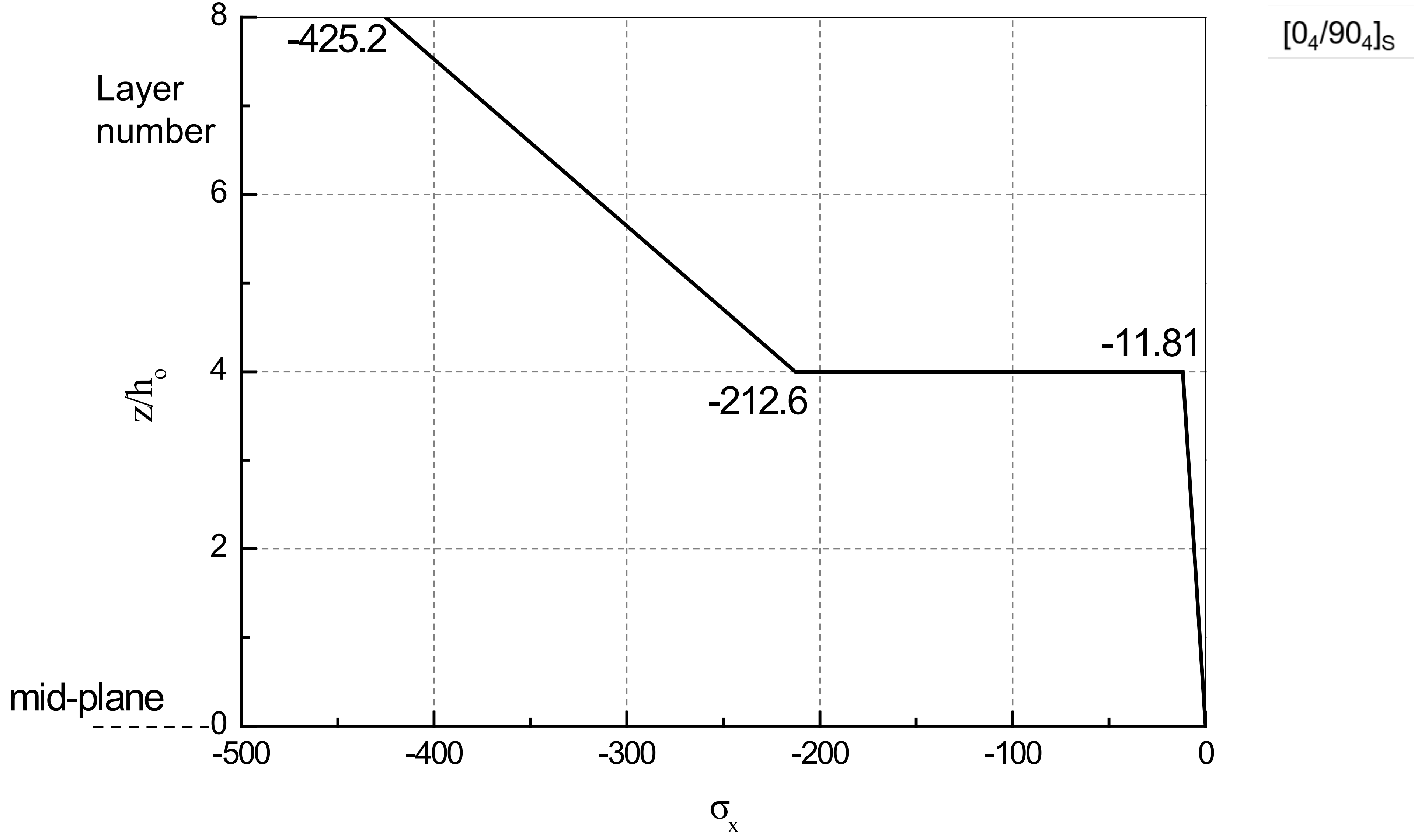
$$k_x = -2.34\ \mathrm{m}^{-1}$$
$$k_y = 0.212\ \mathrm{m}^{-1}$$
$$k_s = 0$$

Allows the calculation of the stresses at the layers of 0º and 90º:

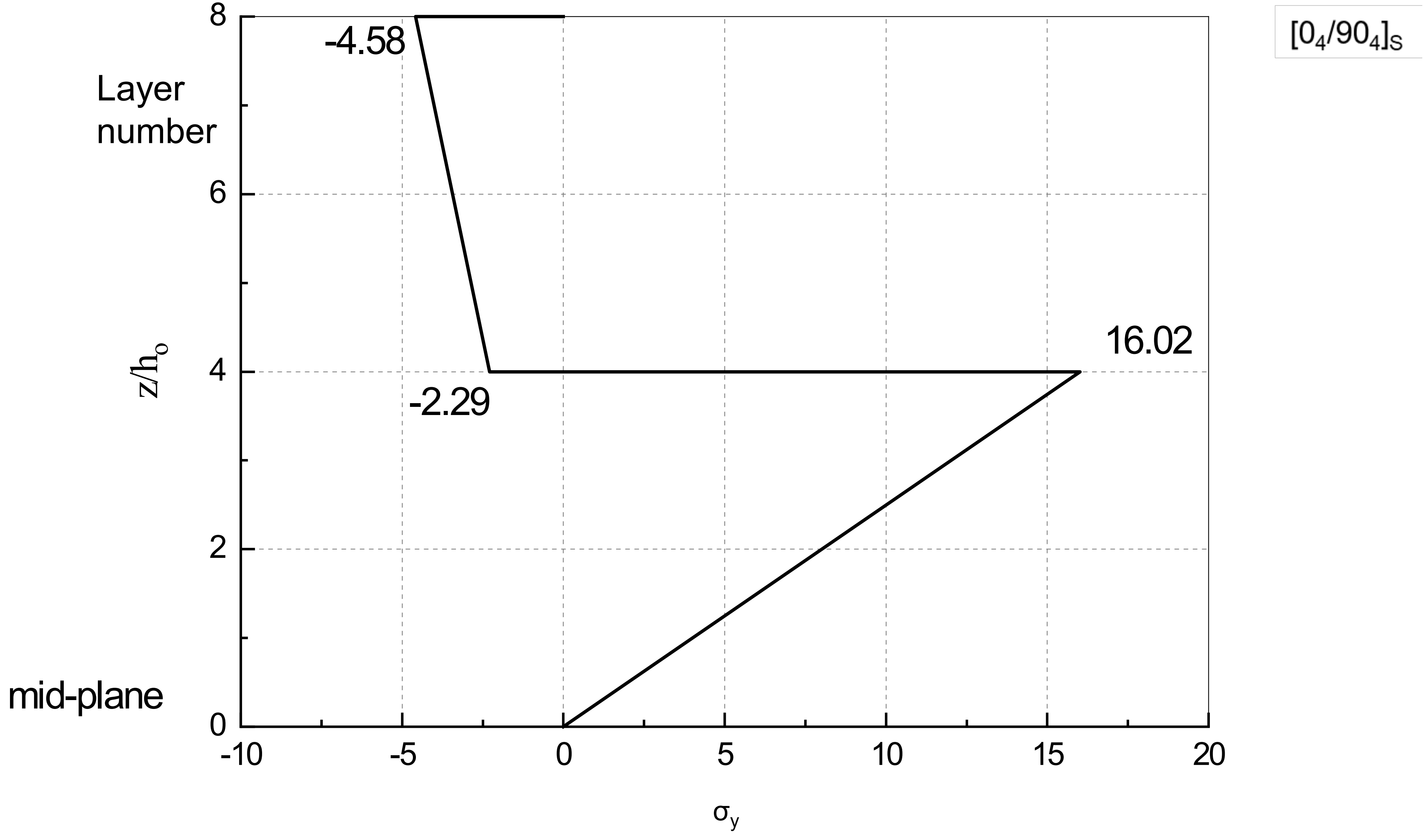
$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_s \end{Bmatrix}^k = z \begin{bmatrix} \overline{Q}_{xx} & \overline{Q}_{xy} & \cancel{\overline{Q}_{xs}} \\ \overline{Q}_{yx} & \overline{Q}_{yy} & \cancel{\overline{Q}_{ys}} \\ \cancel{\overline{Q}_{sx}} & \cancel{\overline{Q}_{sy}} & \overline{Q}_{ss} \end{bmatrix}^k \begin{Bmatrix} k_x \\ k_y \\ \cancel{k_s} \end{Bmatrix}$$

$$Q_{11} = \frac{E_1}{1-\nu_{21}\nu_{12}}$$
$$Q_{12} = \frac{\nu_{12}E_2}{1-\nu_{21}\nu_{12}}$$
$$Q_{22} = \frac{E_2}{1-\nu_{21}\nu_{12}}$$
$$Q_{66} = G_{12}$$

$$\overline{Q}_{xx} = Q_{11}m^4 + 2\left(Q_{12} + 2Q_{66}\right)m^2n^2 + Q_{22}n^4$$
$$\overline{Q}_{xy} = \left(Q_{11} + Q_{22} - 4Q_{66}\right)m^2n^2 + Q_{12}\left(m^4 + n^4\right)$$
$$\overline{Q}_{yy} = Q_{11}n^4 + 2\left(Q_{12} + 2Q_{66}\right)m^2n^2 + Q_{22}m^4$$
$$\overline{Q}_{xs} = \left(Q_{11} - Q_{12} - 2Q_{66}\right)nm^3 + \left(Q_{12} - Q_{22} + 2Q_{66}\right)n^3m$$
$$\overline{Q}_{ys} = \left(Q_{11} - Q_{12} - 2Q_{66}\right)mn^3 + \left(Q_{12} - Q_{22} + 2Q_{66}\right)m^3n$$
$$\overline{Q}_{ss} = \left(Q_{11} + Q_{22} - 2Q_{12} - 2Q_{66}\right)n^2m^2 + Q_{66}\left(n^4 + m^4\right)$$

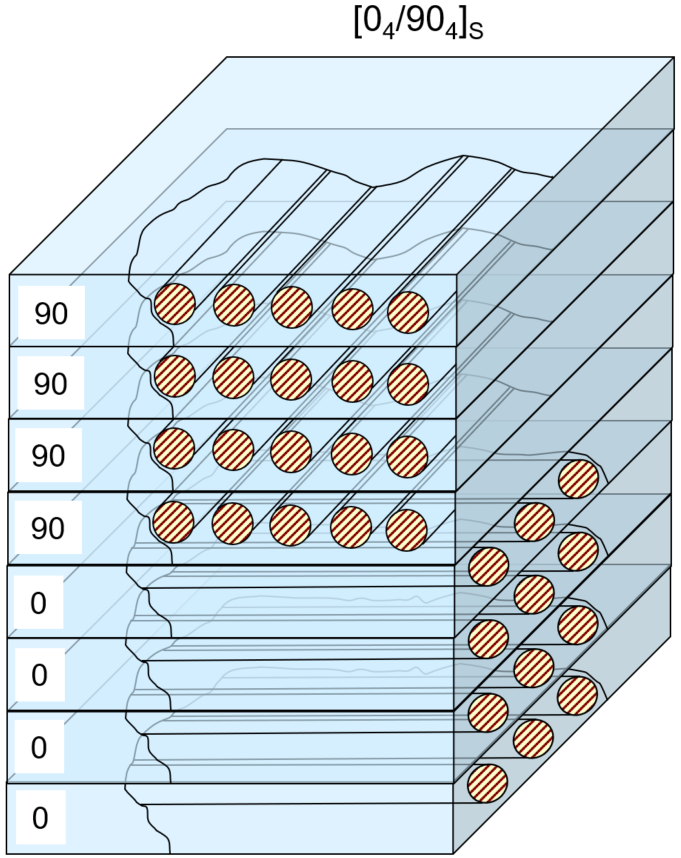


[0$_4$/90$_4$]$_S$
-425.2
Layer number
-11.81
-212.6
mid-plane
z/h_o
σ_x
8
6
4
2
0
-500
-400
-300
-200
-100
0

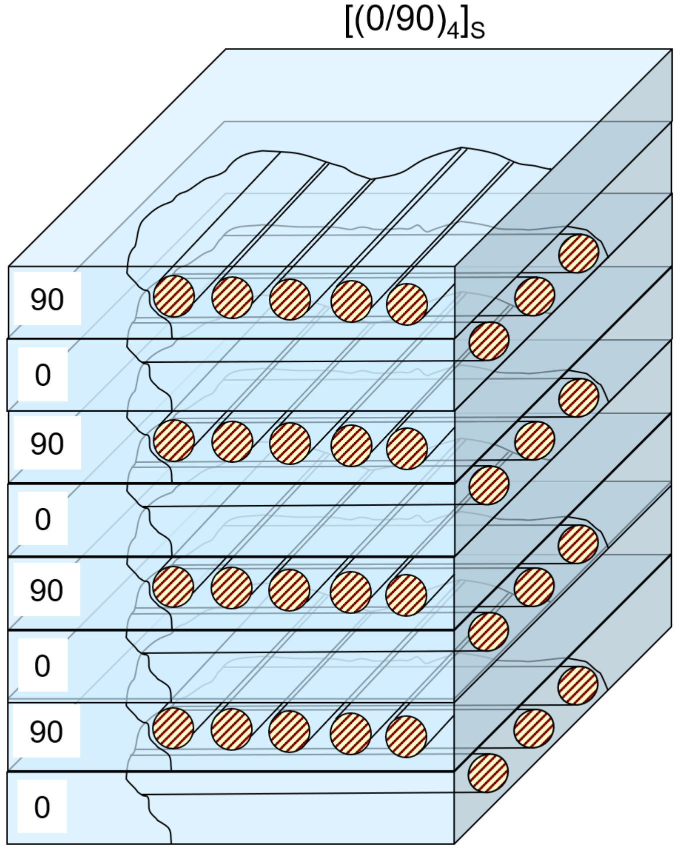


[0$_4$/90$_4$]$_S$
-4.58
Layer number
16.02
-2.29
mid-plane
z/h$_o$
8
6
4
2
0
-10
-5
0
5
10
15
20
σ$_y$

Effect of the stacking sequence on the laminate structural behavior

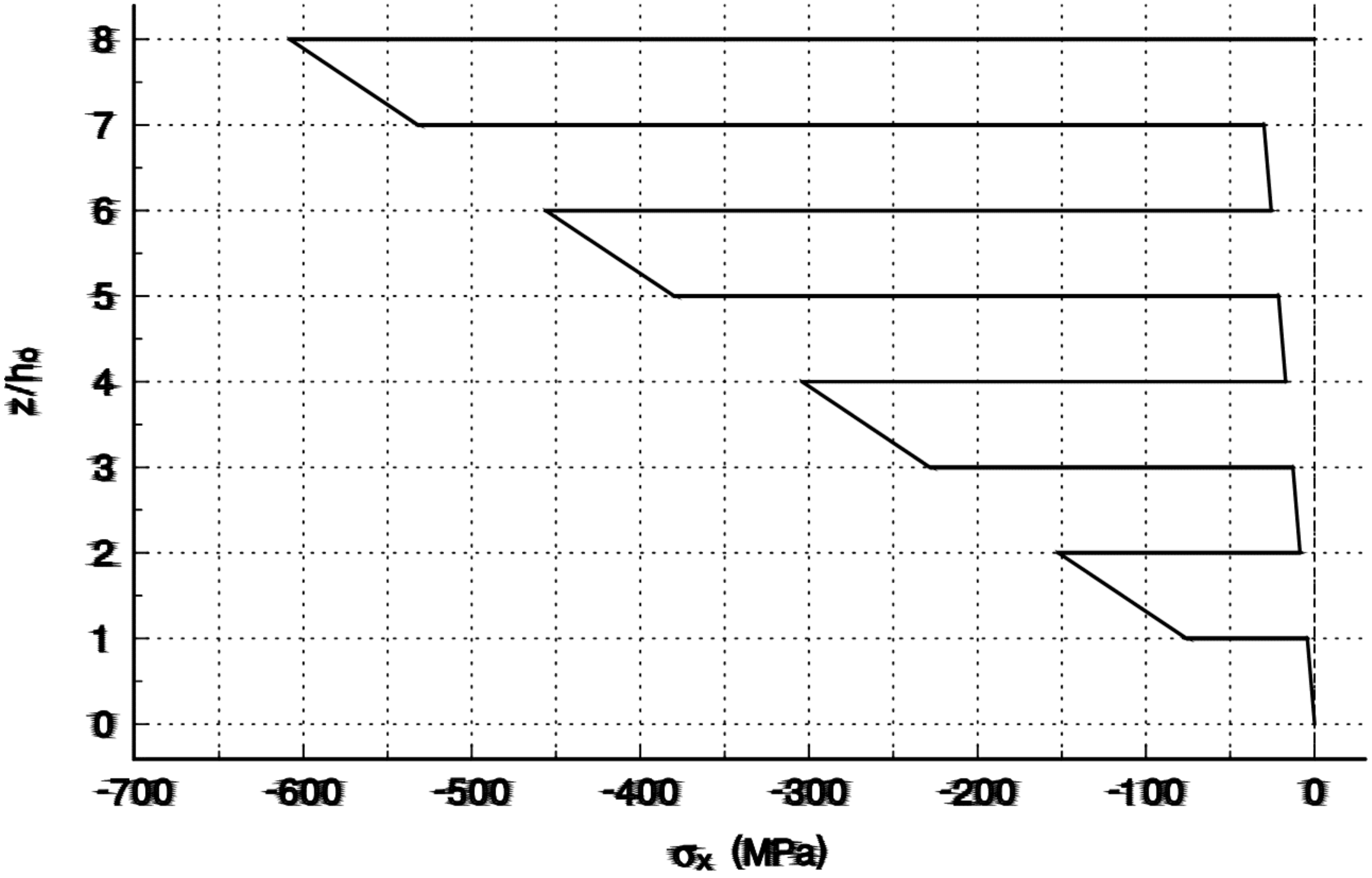


Max deflection at L/2=1.95 10^{-3} m



Max deflection at L/2=2.79 10^{-3} m

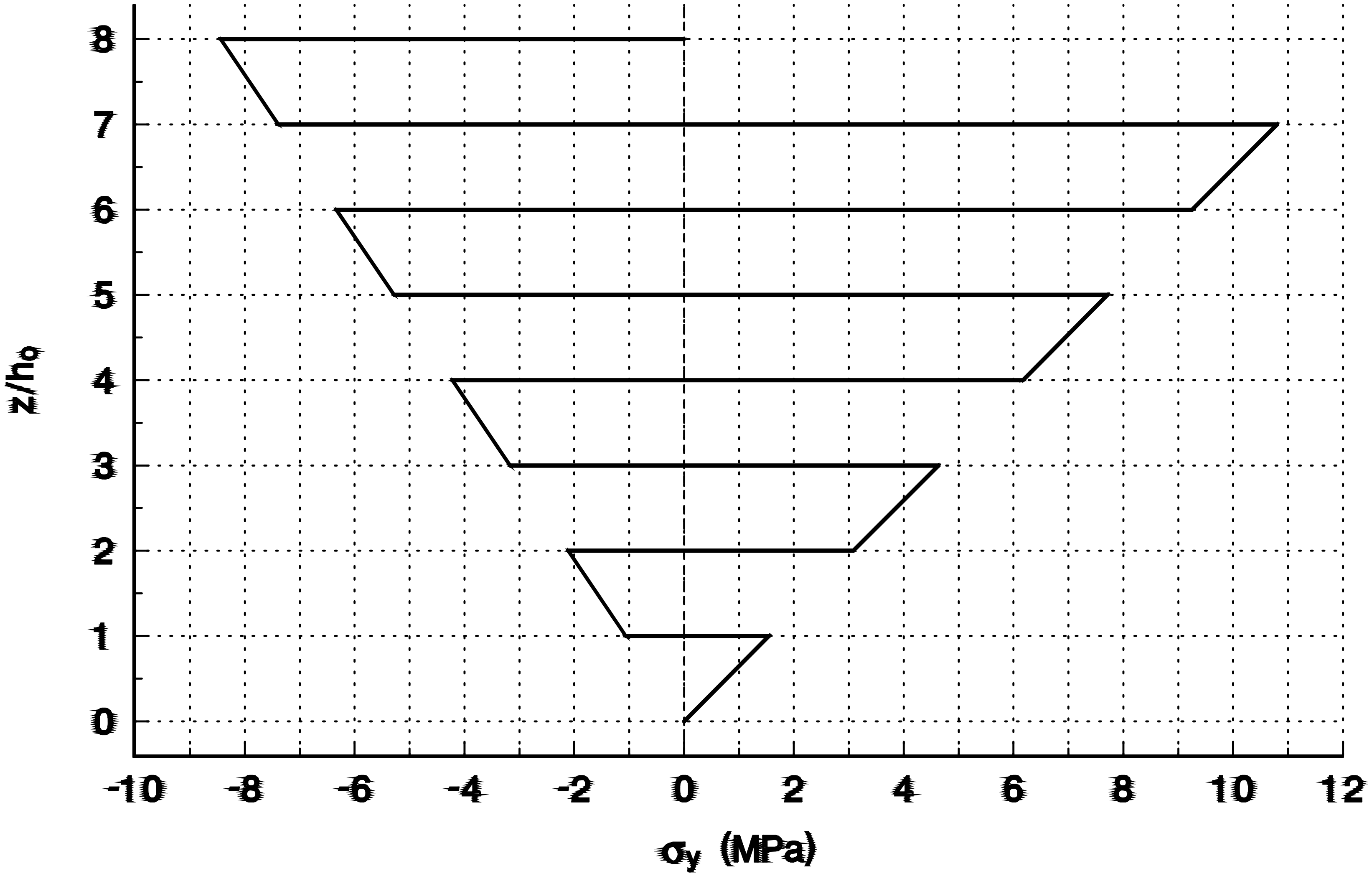
Effect of the stacking sequence on the laminate structural behavior



$[(0/90)_4]_S$

Instead of $[(0_4/90_4)]_S$

Effect of the stacking sequence



$[(0/90)_4]_S$